The Physics of Racing

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Part 1

Weight Transfer

Most autocrossers and race drivers learn early in their careers the importance of balancing a car. Learning to do it consistently and automatically is one essential part of becoming a truly good driver. While the skills for balancing a car are commonly taught in drivers' schools, the rationale behind them is not usually adequately explained. That rationale comes from simple physics. Understanding the physics of driving not only helps one be a better driver, but increases one's enjoyment of driving as well. If you know the deep reasons why you ought to do certain things you will remember the things better and move faster toward complete internalization of the skills.

Balancing a car is controlling weight transfer using throttle, brakes, and steering. This article explains the physics of weight transfer. You will often hear instructors and drivers say that applying the brakes shifts weight to the front of a car and can induce oversteer. Likewise, accelerating shifts weight to the rear, inducing understeer, and cornering shifts weight to the opposite side, unloading the inside tires. But why does weight shift during these maneuvers? How can weight shift when everything is in the car bolted in and strapped down? Briefly, the reason is that inertia acts through the center of gravity (CG) of the car, which is above the ground, but adhesive forces act at ground level through the tire contact patches. The effects of weight transfer are proportional to the height of the CG off the ground. A flatter car, one with a lower CG, handles better and quicker because weight transfer is not so drastic as it is in a high car.

The rest of this article explains how inertia and adhesive forces give rise to weight transfer through Newton's laws. The article begins with the elements and works up to some simple equations that you can use to calculate weight transfer in any car knowing only the wheelbase, the height of the CG, the static weight distribution, and the track, or distance between the tires across the car. These numbers are reported in shop manuals and most journalistic reviews of cars.

Most people remember Newton's laws from school physics. These are fundamental laws that apply to all large things in the universe, such as cars. In the context of our racing application, they are:

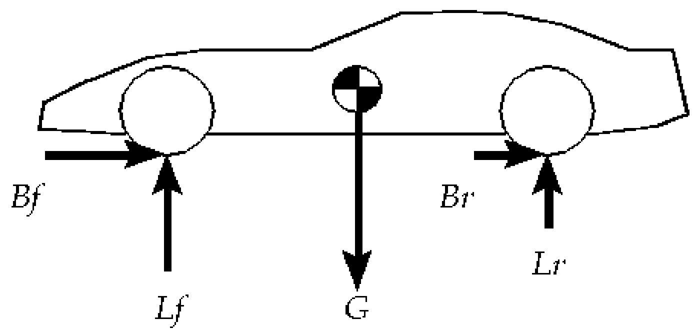
The first law: **a car in straight-line motion at a constant speed**

will keep such motion until acted on by an external force. The only reason a car in neutral will not coast forever is that friction, an external force, gradually slows the car down. Friction comes from the tires on the ground and the air flowing over the car. The tendency of a car to keep moving the way it is moving is the inertia of the car, and this tendency is concentrated at the CG point.

The second law: **When a force is applied to a car, the change in motion is proportional to the force divided by the mass of the car**. This law is expressed by the famous equation $F = ma$, where F is a force, m is the mass of the car, and a is the acceleration, or change in motion, of the car. A larger force causes quicker changes in motion, and a heavier car reacts more slowly to forces. Newton's second law explains why quick cars are powerful and lightweight. The more F and the less m you have, the more a you can get.

The third law: **Every force on a car by another object, such as the ground, is matched by an equal and opposite force on the object by the car**. When you apply the brakes, you cause the tires to push forward against the ground, and the ground pushes back. As long as the tires stay on the car, the ground pushing on them slows the car down.

Let us continue analyzing braking. Weight transfer during accelerating and cornering are mere variations on the theme. We won't consider subtleties such as suspension and tire deflection yet. These effects are very important, but secondary. The figure shows a car and the forces on it during a "one g" braking maneuver. One g means that the total braking force equals the weight of the car, say, in pounds.



A car braking at 1 "gee"

In this figure, the black and white "pie plate" in the center is the CG. G is the force of gravity that pulls the car toward the center of the Earth. This is the weight of the car; weight is just another word for the force of gravity. It is a fact of Nature, only fully explained by Albert Einstein, that gravitational forces act through the CG of an object, just like inertia. This fact can be explained at deeper levels, but such an explanation would take us too far off the subject of weight transfer.

L_f is the lift force exerted by the ground on the front tire, and L_r is the lift force on the rear tire. These lift forces are as real as the ones that keep an airplane in the air, and they keep the car from falling through the ground to the center of the Earth.

We don't often notice the forces that the ground exerts on objects because they are so ordinary, but they are at the essence of car dynamics. The reason is that the magnitude of these forces determine the ability of a tire to stick, and imbalances between the front and rear lift forces account for understeer and oversteer. The figure only shows forces on the car, not forces on the ground and the CG of the Earth. Newton's third law requires that these equal and opposite forces exist, but we are only concerned about how the ground and the Earth's gravity affect the car.

If the car were standing still or coasting, and its weight distribution were 50-50, then L_f would be the same as L_r. It is always the case that L_f plus L_r equals G, the weight of the car. Why? Because of Newton's first law. The car is not changing its motion in the vertical direction, at least as long as it doesn't get airborne, so the total sum of all forces in the vertical direction must be zero. G points down and counteracts the sum of L_f and L_r, which point up.

Braking causes L_f to be greater than L_r. Literally, the "rear end gets light," as one often hears racers say. Consider the front and rear braking forces, B_f and B_r, in the diagram. They push backwards on the tires, which push on the wheels, which push on the suspension parts, which push on the rest of the car, slowing it down. But these forces are acting at ground level, not at the level of the CG. The braking forces are indirectly slowing down the car by pushing at ground level, while the inertia of the car is 'trying' to keep it moving forward as a unit at the CG level.

The braking forces create a rotating tendency, or torque, about the CG. Imagine pulling a table cloth out from under some glasses and candelabra. These objects would have a tendency to tip or rotate over, and the tendency is greater for taller objects and is greater the harder you pull on the cloth. The rotational tendency of a car under braking is due to identical physics.

The braking torque acts in such a way as to put the car up on its nose. Since the car does not actually go up on its nose (we hope), some other forces must be counteracting that tendency, by Newton's first law. G cannot be doing it since it passes right through the cetner of gravity. The only forces that can counteract that tendency are the lift forces, and the only way they can do so is for L_f to become greater than L_r. Literally, the ground pushes up harder on the front tires during braking to try to keep the car from tipping forward.

By how much does L_f exceed L_r? The braking torque is proportional to the sum of the braking forces and to the height of the CG. Let's say that height is 20 inches. The counterbalancing torque resisting the braking torque is proportional to L_f and half the wheelbase (in a car with 50-50 weight distribution), minus L_r times half the wheelbase since L_r is helping the braking forces upend the car. L_f has a lot of work to do: it must resist the torques of both the braking forces and the lift on the rear tires. Let's say the wheelbase is 100 inches. Since we are braking at one g, the braking forces equal G, say, 3200 pounds. All this is summarized in the following equations:

$$3200\,\text{lbs times } 20\,\text{inches} = L_f \text{ times } 50\,\text{inches} - L_r \text{ times } 50\,\text{inches}$$

$$L_f + L_r = 3200\,\text{lbs (this is always true)}$$

With the help of a little algebra, we can find out that

$$L_f = 1600 + 3200/5 = 2240\,\text{lbs}$$

$$L_r = 1600 - 3200/5 = 960\,\text{lbs}$$

Thus, by braking at one g in our example car, we add 640 pounds of load to the front tires and take 640 pounds off the rears! This is very pronounced weight transfer.

By doing a similar analysis for a more general car with CG height of h, wheelbase w, weight G, static weight distribution d expressed as a fraction of weight in the front, and braking with force B, we can show that

$$L_f = dG + Bh/w$$

$$L_r = (1 - d)G - Bh/w$$

These equations can be used to calculate weight transfer during acceleration by treating acceleration force as negative braking force. If you have acceleration figures in gees, say from a *G-analyst* or other device, just multiply them by the weight of the car to get acceleration forces (Newton's second law!). Weight transfer during cornering can be analyzed in a similar way, where the track of the car replaces the wheelbase and d is always 50% (unless you account for the weight of the driver). Those of you with science or engineering backgrounds may enjoy deriving these equations for yourselves. The equations for a car doing a combination of braking and cornering, as in a trail braking maneuver, are much more complicated and require some mathematical tricks to derive.

Now you know why weight transfer happens. The next topic that comes to mind is the physics of tire adhesion, which explains how weight transfer can lead to understeer and oversteer conditions.

Part 2

Keeping Your Tires Stuck to the Ground

In last month's article, we explained the physics behind weight transfer. That is, we explained why braking shifts weight to the front of the car, accelerating shifts weight to the rear, and cornering shifts weight to the outside of a curve. Weight transfer is a side-effect of the tires keeping the car from flipping over during maneuvers. We found out that a one G braking maneuver in our 3200 pound example car causes 640 pounds to transfer from the rear tires to the front tires. The explanations were given directly in terms of Newton's fundamental laws of Nature.

This month, we investigate what causes tires to stay stuck and what causes them to break away and slide. We will find out that you can make a tire slide either by pushing too hard on it or by causing weight to transfer off the tire by your control inputs of throttle, brakes, and steering. Conversely, you can cause a sliding tire to stick again by pushing less hard on it or by tranferring weight to it. The rest of this article explains all this in term of (you guessed it) physics.

This knowledge, coupled with a good 'instinct' for weight transfer, can help a driver predict the consequences of all his or her actions and develop good instincts for staying out of trouble, getting out of trouble when it comes, and driving consistently at ten tenths. It is said of Tazio Nuvolari, one of the greatest racing drivers ever, that he knew at all times while driving the weight on each of the four tires to within a few pounds. He could think, while driving, how the loads would change if he lifted off the throttle or turned the wheel a little more, for example. His knowledge of the physics of racing enabled him to make tiny, accurate adjustments to suit every circumstance, and perhaps to make these adjustments better than his competitors. Of course, he had a very fast brain and phenomenal reflexes, too.

I am going to ask you to do a few physics "lab" experiments with me to investigate tire adhesion. You can actually do them, or you can just follow along in your imagination. First, get a tire and wheel off your car. If you are a serious autocrosser, you probably have a few loose sets in your garage. You can do the experiments with a heavy box or some object that is easier to handle than a tire, but the numbers you get won't apply directly to tires, although the principles we investigate will apply.

Weigh yourself both holding the wheel and not holding it on a bathroom scale. The difference is the weight of the tire and wheel assembly. In my case, it is 50 pounds (it would be a lot less if I had those $3000 Jongbloed wheels! Any sponsors reading?). Now put the wheel on the ground or on a table and push sideways with your hand against the tire until it slides. When you push it, push down low near the point where the tire touches the ground so it doesn't tip over.

The question is, how hard did you have to push to make the tire slide? You can find out by putting the bathroom scale between your hand and the tire when you push. This procedure doesn't give a very accurate reading of the force you need to make the tire slide, but it gives a rough estimate. In my case, on the concrete walkway in front of my house, I had to push with 85 pounds of force (my neighbors don't bother staring at me any more; they're used to my strange antics). On my linoleum kitchen floor, I only had to push with 60 pounds (but my wife does stare at me when I do this stuff in the house). What do these numbers mean?

They mean that, on concrete, my tire gave me $85/50 = 1.70$ gees of sideways resistance before sliding. On a linoleum race course (ahem!), I would only be able to get $60/50 = 1.20G$. We have directly experienced the physics of grip with our bare hands. The fact that the tire resists sliding, up to a point, is called the *grip phenomenon*. If you could view the interface between the ground and the tire with a microscope, you would see complex interactions between long-chain rubber molecules bending, stretching, and locking into concrete molecules creating the grip. Tire researchers look into the detailed workings of tires at these levels of detail.

Now, I'm not getting too excited about being able to achieve $1.70G$ cornering in an autocross. Before I performed this experiment, I frankly expected to see a number below $1G$. This rather unbelievable number of $1.70G$ would certainly not be attainable under driving conditions, but is still a testimony to the rather unbelievable state of tire technology nowadays. Thirty years ago, engineers believed that one G was theoretically impossible from a tire. This had all kinds of consequences. It implied, for example, that dragsters could not possibly go faster than 200 miles per hour in a quarter mile: you can go $\sqrt{2ax} = 198.48$ mph if you can keep $1G$ acceleration all the way down the track. Nowadays, drag racing safety watchdogs are working hard to keep the cars under 300 mph; top fuel dragsters launch at more than

3 gees.

For the second experiment, try weighing down your tire with some ballast. I used a couple of dumbells slung through the center of the wheel with rope to give me a total weight of 90 pounds. Now, I had to push with 150 pounds of force to move the tire sideways on concrete. Still about $1.70G$. We observe the fundamental law of adhesion: the force required to slide a tire is proportional to the weight supported by the tire. When your tire is on the car, weighed down with the car, you cannot push it sideways simply because you can't push hard enough.

The force required to slide a tire is called the *adhesive limit* of the tire, or sometimes the *stiction*, which is a slang combination of "stick" and "friction." This law, in mathematical form, is

$$F \leq \mu W$$

where F is the force with which the tire resists sliding; μ is the *coefficient of static friction* or *coefficient of adhesion*; and W is the weight or vertical load on the tire contact patch. Both F and W have the units of force (remember that weight is the force of gravity), so μ is just a number, a proportionality constant. This equation states that the sideways force a tire can withstand before sliding is less than or equal to μ times W. Thus, μW is the maximum sideways force the tire can withstand and is equal to the stiction. We often like to speak of the sideways acceleration the car can achieve, and we can convert the stiction force into acceleration in gees by dividing by W, the weight of the car. μ can thus be measured in gees.

The coefficient of static friction is not exactly a constant. Under driving conditions, many effects come into play that reduce the stiction of a good autocross tire to somewhere around $1.10G$. These effects are deflection of the tire, suspension movement, temperature, inflation pressure, and so on. But the proportionality law still holds reasonably true under these conditions. Now you can see that if you are cornering, braking, or accelerating at the limit, which means at the adhesive limit of the tires, any weight transfer will cause the tires unloaded by the weight transfer to pass from sticking into sliding.

Actually, the transition from sticking 'mode' to sliding mode should not be very abrupt in a well-designed tire. When one speaks of a "forgiving" tire, one means a tire that breaks away slowly as it gets more and more force or less and less weight, giving the driver time to correct. Old, hard tires are, generally speaking, less forgiving than new, soft tires. Low-profile tires are less forgiving than high-profile tires. Slicks are less forgiving than DOT tires. But these are very broad generalities and tires must be judged individually, usually by getting some word-of-mouth recommendations or just by trying them out in an autocross. Some tires are so unforgiving that they break away virtually without warning, leading to driver dramatics usually resulting in a spin. Forgiving tires are much easier to control and much more fun to drive with.

"Driving by the seat of your pants" means sensing the slight changes in cornering, braking, and acceleration forces that signal that one or more tires are about to slide. You can sense these change literally in your seat, but you can also feel changes in steering resistance and in the sounds the tires make. Generally, tires 'squeak' when they are nearing the limit, 'squeal' at the limit, and 'squall' over the limit. I find tire sounds very informative and always listen to them while driving.

So, to keep your tires stuck to the ground, be aware that accelerating gives the front tires less stiction and the rear tires more, that braking gives the front tire more stiction and the rear tires less, and that cornering gives the inside tires less stiction and the outside tires more. These facts are due to the combination of weight transfer and the grip phenomenon. Finally, drive smoothly, that is, translate your awareness into gentle control inputs that always keep appropriate tires stuck at the right times. This is the essential knowledge required for car control, and, of course, is much easier said than done. Later articles will use the knowledge we have accumulated so far to explain understeer, oversteer, and chassis set-up.

Part 3

Basic Calculations

In the last two articles, we plunged right into some relatively complex issues, namely weight transfer and tire adhesion. This month, we regroup and review some of the basic units and dimensions needed to do dynamical calculations. Eventually, we can work up to equations sufficient for a full-blown computer simulation of car dynamics. The equations can then be 'doctored' so that the computer simulation will run fast enough to be the core of an autocross computer game. Eventually, we might direct this series of articles to show how to build such a game in a typical microcomputer programming language such as C or BASIC, or perhaps even my personal favorite, LISP. All of this is in keeping with the spirit of the series, the Physics of Racing, because so much of physics today involves computing. Software design and programming are essential skills of the modern physicist, so much so that many of us become involved in computing full time.

Physics is the science of measurement. Perhaps you have heard of highly abstract branches of physics such as quantum mechanics and relativity, in which exotic mathematics is in the forefront. But when theories are taken to the laboratory (or the race course) for testing, all the mathematics must boil down to quantities that can be measured. In racing, the fundamental quantities are distance, time, and mass. This month, we will review basic equations that will enable you to do quick calculations in your head while cooling off between runs. It is very valuable to develop a skill for estimating quantities quickly, and I will show you how.

Equations that don't involve mass are called *kinematic*. The first kinematic equation relates speed, time, and distance. If a car is moving at a constant speed or velocity, v, then the distance d it travels in time t is

$$d = vt$$

or velocity times time. This equation really expresses nothing more than the definition of velocity.

If we are to do mental calculations, the first hurdle we must jump comes from the fact that we usually measure speed in miles per hour (mph), but distance in feet and time in seconds. So, we must modify our equation with a conversion factor, like this

$$d\,(\text{feet}) = v\frac{\text{miles}}{\text{hour}}\,t\,(\text{seconds})\frac{5280\,\text{feet/mile}}{3600\,\text{seconds/hour}}$$

If you "cancel out" the units parts of this equation, you will see that you get feet on both the left and right hand sides, as is appropriate, since equality is required of any equation. The conversion factor is 5280/3600, which happens to equal 22/15. Let's do a few quick examples. How far does a car go in one second (remember, say, "one-one-thousand, two-one-thousand," *etc.* to yourself to count off seconds)? At fifteen mph, we can see that we go

$$d = 15\,\text{mph times 1 sec times } 22/15 = 22\text{ feet}$$

or about 1 and a half car lengths for a 14 and 2/3 foot car like a late-model Corvette. So, at 30 mph, a second is three car lengths and at 60 mph it is six. If you lose an autocross by 1 second (and you'll be pretty good if you can do that with all the good drivers in our region), you're losing by somewhere between 3 and 6 car lengths! This is because the average speed in an autocross is between 30 and 60 mph.

Everytime you plow a little or get a little sideways, just visualize your competition overtaking you by a car length or so. One of the reasons autocross is such a difficult sport, but also such a pure sport, from the driver's standpoint, is that you can't make up this time. If you blow a corner in a road race, you may have a few laps in which to make it up. But to win an autocross against good competition, you must drive nearly perfectly. The driver who makes the fewest mistakes usually wins!

The next kinematic equation involves acceleration. It so happens that the distance covered by a car at constant acceleration from a standing start is given by

$$d = \frac{1}{2}at^2$$

or 1/2 times the acceleration times the time, squared. What conversions will help us do mental calculations with this equation? Usually, we like to measure acceleration in Gs. One G happens to be 32.1 feet per second squared. Fortunately, we don't have to deal with miles and hours here, so our equation becomes,

$$d\,(\text{feet}) = 16a\,(G\text{s})\,t\,(\text{seconds})^2$$

roughly. So, a car accelerating from a standing start at $\frac{1}{2}G$, which is a typical number for a good, stock sports car, will go 8 feet in 1 second. Not very far! However, this picks up rapidly. In two seconds, the car will go 32 feet, or over two car lengths.

Just to prove to you that this isn't crazy, let's answer the question "How long will it take a car accelerating at $\frac{1}{2}G$ to do the quarter mile?" We invert the equation above (recall your high school algebra), to get

$$t = \sqrt{d\,(\text{feet})\ 16a\,(G\text{s})}$$

and we plug in the numbers: the quarter mile equals 1320 feet, $a = \frac{1}{2}G$, and we get $t = \sqrt{1320/8} = \sqrt{165}$ which is about 13 seconds. Not too unreasonable! A real car will not be able to keep up full $\frac{1}{2}G$ acceleration for a quarter mile due to air resistance and reduced torque in the higher gears. This explains why real (stock) sports cars do the quarter mile in 14 or 15 seconds.

The more interesting result is the fact that it takes a full second to go the first 8 feet. So, we can see that the launch is critical in an autocross. With excessive wheelspin, which robs you of acceleration, you can lose a whole second right at the start. Just visualize your competition pulling 8 feet ahead instantly, and that margin grows because they are 'hooked up' better.

For doing these mental calculations, it is helpful to memorize a few squares. 8 squared is 64, 10 squared is 100, 11 squared is 121, 12 squared is 144, 13 squared is 169, and so on. You can then estimate square roots in your head with acceptable precision.

Finally, let's examine how engine torque becomes force at the drive wheels and finally acceleration. For this examination, we will need to know the mass of the car. Any equation in physics that involves mass is called *dynamic*, as opposed to kinematic. Let's say we have a Corvette that weighs 3200 pounds and produces 330 foot-pounds of torque at the crankshaft. The Corvette's automatic transmission has a first gear ratio of 3.06 (the auto is the trick set up for vettes—just ask Roger Johnson or Mark Thornton). A transmission is nothing but a set of circular, rotating levers, and the gear ratio is the leverage, multiplying the torque of the engine. So, at the output of the transmission, we have

$$3.06 \times 330 = 1010\,\text{foot-pounds}$$

of torque. The differential is a further lever-multiplier, in the case of the Corvette by a factor of 3.07, yielding 3100 foot pounds at the center of the rear wheels (this is a lot of torque!). The distance from the center of the wheel to the ground is about 13 inches, or 1.08 feet, so the maximum force that the engine can put to the ground in a rearward direction (causing the ground to push back forward—remember part 1 of this series!) in first gear is

$$3100\,\text{foot-pounds}/1.08\,\text{feet} = 2870\,\text{pounds}$$

Now, at rest, the car has about 50/50 weight distribution, so there is about 1600 pounds of load on the rear tires. You will remember from last month's article on tire adhesion that the tires cannot respond with a forward force much greater than the weight that is on them, so they simply will spin if you stomp on the throttle, asking them to give you 2870 pounds of force.

We can now see why it is important to squeeeeeeeze the throttle gently when launching. In the very first instant of a launch, your goal as a driver is to get the engine up to where it is pushing on the tire contact patch at about 1600 pounds. The tires will squeal or hiss just a little when you get this right. Not so coincidentally, this will give you a forward force of about 1600 pounds, for an $F = ma$ (part 1) acceleration of about $\frac{1}{2}G$, or half the weight of the car. The main reason a car will accelerate with only $\frac{1}{2}G$ to start with is that half of the weight is on the front wheels and is unavailable to increase the stiction of the rear, driving tires. Immediately, however, there will be some weight transfer to the rear. Remembering part 1 of this series again, you can estimate that about 320 pounds will be transferred to the rear immediately. You can now ask the tires to give you a little more, and you can gently push on the throttle. Within a second or so, you can be at full throttle, putting all that torque to work for a beautiful hole shot!

In a rear drive car, weight transfer acts to make the driving wheels capable of withstanding greater forward loads. In a front drive car, weight transfer works against acceleration, so you have to be even more gentle on the throttle if you have a lot of power. An all-wheel drive car puts all the wheels to work delivering force to the ground and is theoretically the best.

Technical people call this style of calculating "back of the envelope," which is a somewhat picturesque reference to the habit we have of writing equations and numbers on any piece of paper that happens to be handy. You do it without calculators or slide rules or abacuses. You do it in the garage or the pits. It is not exactly precise, but gives you a rough idea, say within

10 or 20 percent, of the forces and accelerations at work. And now you know how to do back-of-the-envelope calculations, too.

Part 4

There Is No Such Thing as Centrifugal Force

One often hears of "centrifugal force." This is the apparent force that throws you to the outside of a turn during cornering. If there is anything loose in the car, it will immediately slide to the right in a left hand turn, and *vice versa*. Perhaps you have experienced what happened to me once. I had omitted to remove an empty Pepsi can hidden under the passenger seat. During a particularly aggressive run (something for which I am not unknown), this can came loose, fluttered around the cockpit for a while, and eventually flew out the passenger window in the middle of a hard left hand corner.

I shall attempt to convince you, in this month's article, that centrifugal force is a fiction, and a consequence of the fact first noticed just over three hundred years ago by Newton that objects tend to continue moving in a straight line unless acted on by an external force.

When you turn the steering wheel, you are trying to get the front tires to push a little sideways on the ground, which then pushes back, by Newton's third law. When the ground pushes back, it causes a little sideways acceleration. This sideways acceleration is a change in the sideways velocity. The acceleration is proportional to the sideways force, and inversely proportional to the mass of the car, by Newton's second law. The sideways acceleration thus causes the car to veer a little sideways, which is what you wanted when you turned the wheel. If you keep the steering and throttle at constant positions, you will continue to go mostly forwards and a little sideways until you end up where you started. In other words, you will go in a circle. When driving through a sweeper, you are going part way around a circle. If you take skid pad lessons (highly recommended), you will go around in circles all day.

If you turn the steering wheel a little more, you will go in a tighter circle, and the sideways force needed to keep you going is greater. If you go around the same circle but faster, the necessary force is greater. If you try to go around too fast, the adhesive limit of the tires will be exceeded, they will slide, and you will not stick to the circular path—you will not "make it."

From the discussion above, we can see that in order to turn right, for example, a force, pointing to the right, must act on the car that veers it away from the straight line it naturally tries to follow. If the force stays constant, the car will go in a circle. From the point of view of the car, the force always points to the right. From a point of view outside the car, at rest with respect to the ground, however, the force points toward the center of the circle. From this point of view, although the force is constant in *magnitude*, it changes *direction*, going around and around as the car turns, always pointing at the geometrical center of the circle. This force is called *centripetal*, from the Greek for "center seeking." The point of view on the ground is privileged, since objects at rest from this point of view feel no net forces. Physicists call this special point of view an *inertial frame of reference*. The forces measured in an inertial frame are, in a sense, more correct than those measured by a physicist riding in the car. Forces measured inside the car are *biased* by the centripetal force.

Inside the car, all objects, such as the driver, feel the natural inertial tendency to continue moving in a straight line. The driver receives a centripetal force from the car through the seat and the belts. If you don't have good restraints, you may find yourself pushing with your knee against the door and tugging on the controls in order to get the centripetal force you need to go in a circle with the car. It took me a long time to overcome the habit of tugging on the car in order to stay put in it. I used to come home with bruises on my left knee from pushing hard against the door during an autocross. I found that a tight five- point harness helped me to overcome this unnecessary habit. With it, I no longer think about body position while driving—I can concentrate on trying to be smooth and fast. As a result, I use the wheel and the gearshift lever for steering and shifting rather than for helping me stay put in the car!

The 'forces' that the driver and other objects inside the car feel are actually centripetal. The term *centrifugal*, or "center fleeing," refers to the inertial tendency to resist the centripetal force and to continue going straight. If the centripetal force is constant in magnitude, the centrifugal tendency will be constant. There is no such thing as centrifugal force (although it is a convenient fiction for the purpose of some calculations).

Let's figure out exactly how much sideways acceleration is needed to keep a car going at speed v in a circle of radius r. We can then convert this into force using Newton's second law, and then figure out how fast we can go in a circle before exceeding the adhesive limit—in other words, we can derive maximum cornering speed. For the following discussion, it will be helpful for you to draw little back-of-the-envelope pictures (I'm leaving them out, giving our editor a rest from transcribing my graphics into the newsletter).

Consider a very short interval of time, far less than a second. Call it dt (d stands for "delta," a Greek letter mathematicians use as shorthand for "tiny increment"). In time dt, let us say we go forward a distance dx and sideways a distance ds. The forward component of the velocity of the car is approximately $v = dx/dt$. At the beginning of the time interval dt, the car has no sideways velocity. At the end, it has sideways velocity ds/dt. In the time dt, the car has thus had a change in sideways velocity of ds/dt. Acceleration is, precisely, the change in velocity over a certain time, divided by the time; just as velocity is the change in position over a certain time, divided by the time. Thus, the sideways acceleration is

$$a = \frac{ds}{dt}\frac{1}{dt}$$

How is ds related to r, the radius of the circle? If we go forward by a fraction f of the radius of the circle, we must go sideways by exactly the same fraction of dx to stay on the circle. This means that $ds = f\ dx$. The fraction f is, however, nothing but dx/r. By this reasoning, we get the relation

$$ds = dx\frac{dx}{r}$$

We can substitute this expression for ds into the expression for a, and remembering that $v = dx/dt$, we get the final result

$$a = \frac{ds}{dt}\frac{1}{dt} = \frac{dx}{dt}\frac{dx}{dt}\frac{1}{r} = \frac{v^2}{r}$$

This equation simply says quantitatively what we wrote before: that the acceleration (and the force) needed to keep to a circular line increases with the velocity and increases as the radius gets smaller.

What was *not* appreciated before we went through this derivation is that the necessary acceleration increases as the *square* of the velocity. This means that the centripetal force your tires must give you for you to make it through a sweeper is very sensitive to your speed. If you go just a little bit too fast, you might as well go *much* too fast—your're not going to make it. The following table shows the maximum speed that can be achieved in turns of various radii for various sideways accelerations. This table shows the value of the expression

$$\frac{15}{22}\sqrt{32.1a\ (\text{gees})\ r(\text{feet})}$$

which is the solution of $a = v^2/r$ for v, the velocity. The conversion factor 15/22 converts v from feet per second to miles per hour, and 32.1 converts a from gees to feet per second squared. We covered these conversion factors in part 3 of this series.

TABLE 1: SPEED (MILES PER HOUR)

ACCELERATION	RADIUS (FEET)				
(GEES)	50.00	100.00	150.00	200.00	500.00
0.25	13.66	19.31	23.66	27.32	43.19
0.50	19.31	27.32	33.45	38.63	61.08
0.75	23.66	33.45	40.97	47.31	74.81
1.00	27.32	38.63	47.31	54.63	86.38
1.25	30.54	43.19	52.90	61.08	96.57
1.50	33.45	47.31	57.94	66.91	105.79
1.75	36.13	51.10	62.59	72.27	114.27
2.00	38.63	54.63	66.91	77.26	122.16

For autocrossing, the columns for 50 and 100 feet and the row for 1.00 G are most germane. The table tells us that to achieve 1.00 G sideways acceleration in a corner of 50 foot radius (this kind of corner is all too common in autocross), a driver must not go faster than 27.32 miles per hour. To go 30 mph, 1.25 G is required, which is probably not within the capability of an autocross tire at this speed. There is not much subjective difference between 27 and 30 mph, but the objective difference is usually between making a controlled run and spinning badly.

The absolute fastest way to go through a corner is to be just over the limit near the exit, in a controlled slide. To do this, however, you must be pointed in just such a way that when the car breaks loose and slides to the exit of the corner it will be pointed straight down the optimal racing line at the exit when it "hooks up" again. You can smoothly add throttle during this maneuver and be really moving out of the corner. But you must do it smoothly. It takes a long time to learn this, and probably a lifetime to perfect it, but it feels absolutely triumphal when done right. I have not figured out how to drive through a sweeper, except for the exit, at anything greater than the limiting velocity because sweepers are just too long to slide around. If anyone (Ayrton Senna, perhaps?) knows how, please tell me!

The chain of reasoning we have just gone through was first discovered by Newton and Leibniz, working independently. It is, in fact, a derivation in differential calculus, the mathematics of very small quantities. Newton keeps popping up. He was perhaps the greatest of all physicists, having discovered the laws of motion, the law of gravity, and calculus, among other things such as the fact that white light is made up of multiple colors mixed together.

It is an excellent diagnostic exercise to drive a car around a circle marked with cones or chalk and gently to increase the speed until the car slides. If the front breaks away first, your car has natural understeer, and if the rear slides first, it has natural oversteer. You can use this information for chassis tuning. Of course, this is only to be done in safe circumstances, on a rented skid pad or your own private parking lot. The police will gleefully give you a ticket if they catch you doing this in the wrong places.

Part 5
Introduction to the Racing Line

This month, we analyze the best way to go through a corner. “Best” means in the least time, at the greatest average speed. We ask “what is the shape of the driving line through the corner that gives the best time?” and “what are the times for some other lines, say hugging the outside or the inside of the corner?” Given the answers to these questions, we go on to ask “what shape does a corner have to be before the driving line I choose doesn’t make any time difference?” The answer is a little surprising.

The analysis presented here is the simplest I could come up with, and yet is still quite complicated. My calculations went through about thirty steps before I got the answer. Don’t worry, I won’t drag you through the mathematics; I just sketch out the analysis, trying to focus on the basic principles. Anyone who would read through thirty formulas would probably just as soon derive them for him or herself.

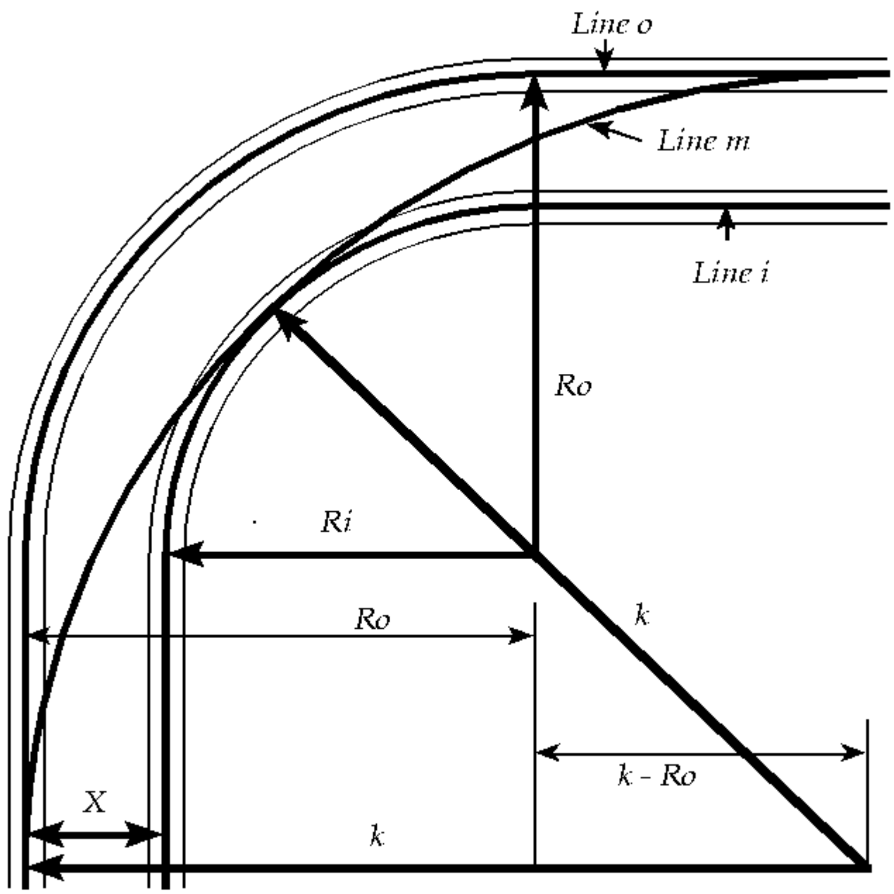
There are several simplifying assumptions I make to get through the analysis. First of all, I consider the corner in isolation; as an abstract entity lifted out of the rest of a course. The actual best driving line through a corner depends on what comes before it and after it. You usually want to optimize exit speed if the corner leads onto a straight. You might not apex if another corner is coming up. You may be forced into an unfavorable entrance by a prior curve or slalom.

Speaking of road courses, you will hear drivers say things like “you have to do such-and-such in turn six to be on line for turn ten and the front straight.” In other words, actions in any one spot carry consequences pretty much all the way around. The ultimate drivers figure out the line for the entire course and drive it as a unit, taking a Zen-like approach. When learning, it is probably best to start out optimizing each kind of corner in isolation, then work up to combinations of two corners, three corners, and so on. In my own driving, there are certain kinds of three corner combinations I know, but mostly I work in twos. I have a long way to go.

It is not feasible to analyze an actual course in an exact, mathematical way. In other words, although science can provide general principles and hints, finding the line is, in practice, an art. For me, it is one of the most fun parts of racing.

Other simplifying assumptions I make are that the car can either accelerate, brake, or corner at constant speed, with abrupt transitions between behaviors. Thus, the lines I analyze are splices of accelerating, braking, and cornering phases. A real car can, must, and should do these things in combination and with smooth transitions between phases. It is, in fact, possible to do an exact, mathematical analysis with a more realistic car that transitions smoothly, but it is much more difficult than the splice-type analysis and does not provide enough more quantitative insight to justify its extra complexity for this article.

Our corner is the following ninety-degree right-hander:



This figure actually represents a family of corners with any constant width, any radius, and short straights before and after. First, we go through the entire analysis with a particular corner of 75 foot radius and 30 foot width, then we end up with times for corners of various radii and widths.

Let us define the following parameters:

$$r = \text{radius of corner center line} = 75 \text{ feet}$$

$$W = \text{width of course} = 30 \text{ feet}$$

$$r_o = \text{radius of outer edge} = r + \tfrac{1}{2}W = 90 \text{ feet}$$

$$r_i = \text{radius of inner edge} = r - \tfrac{1}{2}W = 60 \text{ feet}$$

Now, when we drive this corner, we must keep the tires on the course, otherwise we get a lot of cone penalties (or go into the weeds). It is easiest (though not so realistic) to do the analysis considering the path of the center of gravity of the car rather than the paths of each wheel. So, we define an *effective* course, narrower than the real course, down which we may drive the center of the car.

$$w = \text{width of car} = 6 \text{ feet}$$

$$R_o = \text{effective outer radius} = r_o - \tfrac{1}{2}w = 87 \text{ feet}$$

$$R_i = \text{effective inner radius} = r_i + \tfrac{1}{2}w = 63 \text{ feet}$$

$$X = \text{effective width of course} = W - w = 24 \text{ feet}$$

This course is indicated by the labels and the thick radius lines in the figure.

From last month's article, we know that for a fixed centripetal acceleration, the maximum driving speed increases as the square root of the radius. So, if we drive the largest possible circle through the effective corner, starting at the outside of the entrance straight, going all the way to the inside in the middle of the corner (the *apex*), and ending up at the outside of the exit straight, we can corner at the maximum speed. Such a line is shown in the figure as the thick circle labeled "line m." This is a simplified version of the classic racing line through the corner. Line m reaches the apex at the geometrical center of the circle, whereas the classic racing line reaches an apex after the geometrical center—a *late* apex—because it assumes we are accelerating out of the corner and must therefore have a continuously increasing radius in the second half and a slightly tighter radius in the first half to prepare for the acceleration. But, we continue analyzing the geometrically perfect line because it is relatively easy. The figure shows also Line i, the *inside* line, which come up the inside of the entrance straight, corners on the inside, and goes down the inside of the exit straight; and Line o, the *outside* line, which comes up the outside, corners on the outside, and exits on the outside.

One might argue that there are certain advantages of line i over line m. Line i is considerably shorter than Line m, and although we have to go slower through the corner part, we have less total distance to cover and might get through faster. Also, we can accelerate on part of the entrance chute and all the way on the exit chute, while we have to drive line m at constant speed. Let's find out how much time it takes to get through lines i and m. We include line o for completeness, even though it looks bad because it is both slower and longer than m.

If we assume a maximum centripetal acceleration of 1.10g, which is just within the capability of autocross tires, we get the following speeds for the cornering phases of Lines i, o, and m:

Cornering Speed (mph)		
Line i	Line o	Line m
32.16	37.79	48.78
v_i	v_o	v_m

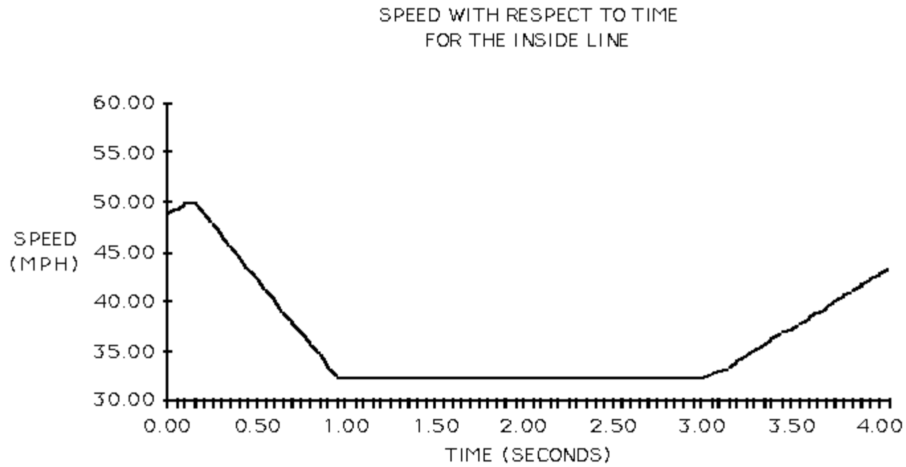
Line m is all cornering, so we can easily calculate the time to drive it once we know the radius, labeled k in the figure. A geometrical analysis results in

$$k = 3.414(R_o - 0.707R_i) = 145\,\text{feet}$$

and the time is

$$t_m = \left(\frac{\pi}{2}k\right) / \left(\frac{22}{15}v_m\right) = 3.18\,\text{seconds.}$$

For line i, we accelerate for a bit, brake until we reach 32.16 mph, corner at that speed, and then accelerate on the exit. Let's assume, to keep the comparison fair, that we have timing lights at the beginning and end of line m and that we can begin driving line i at 48.78 mph, the same speed that we can drive line m. Let us also assume that the car can accelerate at $\frac{1}{2}$g and brake at 1g. Our driving plan for line i results in the following velocity profile:



Because we can begin by accelerating, we start beating line m a little. We have to brake hard to make the corner. Finally, although we accelerate on the exit, we don't quite come up to 48.78 mph, the exit speed for line m. But, we don't care about exit speed, only time through the corner. Using the velocity profile above, we can calculate the time for line i, call it t_i, to be 4.08 seconds. Line i loses by 9/10ths of a second. It is a fair margin to lose an autocross by this much over a whole course, but this analysis shows we can lose it in just one typical corner! In this case, line i is a catastrophic mistake. Incidentally, line o takes 4.24 seconds = t_o.

What if the corner were tighter or of greater radius? The following table shows some times for 30 foot wide corners of various radii:

radius	30.00	45.00	60.00	75.00	90.00	95.00
t_o	3.99	4.06	4.15	4.24	4.35	4.38
t_i	3.94	3.94	4.00	4.08	4.17	4.21
t_m	2.64	2.83	3.01	3.18	3.34	3.39
margin	1.30	1.11	1.01	0.90	0.83	0.82

Line i *never* beats line m even though that as the radius increases, the margin of loss decreases. The trend is intuitive because corners of greater radius are also longer and the extra speed in line m over line i is less. The margin is greatest for tight corners because the width is a greater fraction of the length and the speed differential is greater.

How about for various widths? The following table shows times for a 75 foot radius corner of several widths:

width	10.00	30.00	50.00	70.00	90.00
t_o	2.68	4.24	5.47	6.50	7.41
t_i	2.62	4.08	5.32	6.45	7.51
t_m	2.46	3.18	3.77	4.27	4.73
margin	0.16	0.90	1.55	2.18	2.79

The wider the course, the greater the margin of loss. This is, again, intuitive since on a wide course, line m is a really large circle through even a very tight corner. Note that line o becomes better than line i for wide courses. This is because the speed differential between lines o and i is very great for wide courses. The most notable fact is that line m beats line i by 0.16 seconds even on a course that is only four feet wider than the car! You really must "use up the whole course."

So, the answer is, under the assumptions made, that the inside line is *never* better than the classic racing line. For the splice-type car behavior assumed, I conjecture that *no* line is faster than line m.

We have gone through a simplified kind of *variational analysis*. Variational analysis is used in all branches of physics, especially mechanics and optics. It is possible, in fact, to express all theories of physics, even the most arcane, in variational form, and many physicists find this form very appealing. It is also possible to use variational analysis to write a computer program that finds an approximately perfect line through a complete, realistic course.

Part 6

Speed and Horsepower

The title of this month's article consists of two words dear to every racer's heart. This month, we do some "back of the envelope" calculations to investigate the basic physics of speed and horsepower (the "back of the envelope" style of calculating was covered in part 3 of this series).

How much horsepower does it take to go a certain speed? At first blush, a physicist might be tempted to say "none," because he or she remembers Newton's first law, by which an object moving at a constant speed in a straight line continues so moving forever, even to the end of the Universe, unless acted on by an external force. Everyone knows, however, that it is necessary to keep your foot on the gas to keep a car moving at a constant speed. Keeping your foot on the gas means that you are making the engine apply a backward force to the ground, which applies a reaction force forward on the car, to keep the car moving. In fact, we know a few numbers from our car's shop manual. A late model Corvette, for example, has a top speed of about 150 miles per hour and about 240 hp. This means that if you keep your foot *all* the way down, using up all 240 hp, you can eventually go 150 mph. It takes a while to get there. In this car, you can get to 60 mph in about 6 seconds (if you don't spin the drive wheels), to 100 mph in about 15 seconds, and 150 in about a minute.

All this seems to contradict Newton's first law. What is going on? An automobile moving at constant speed in a straight line on level ground is, in fact, acted on by a number of external forces that tend to slow it down. Without these forces, the car *would* coast forever as guaranteed by Newton's first law. You must counteract these forces with the engine, which indirectly creates a reaction force that keeps the car going. When the car is going at a constant speed, the *net* force on the car, that is, the speeding-up forces minus the slowing-down forces, is zero.

The most important external, slowing-down force is *air resistance* or *drag.* The second most important force is friction between the tires and the ground, the so-called *rolling resistance.* Both these forces are called *resistance* because they always act to oppose the forward motion of the car in whatever direction it is going. Another physical effect that slows a car down is internal friction in the drive train and wheel bearings. Acting internally, these forces cannot slow the car. However, they push backwards on the tires, which push forward on the ground, which pushes back by Newton's third law, slowing the car down. The internal friction forces are opposed by external reaction forces, which act as slight braking forces, slowing the car. So, Newton and the Universe are safe; everything is working as it should.

How big are the resistance forces, and what role does horsepower play? The physics of air resistance is very complex and an area of vigorous research today. Most of this research is done by the aerospace industry, which is technologically very closely related to the automobile industry, especially when it comes to racing. We'll slog through some arithmetic here to come up with a table that shows how much horsepower it takes to sustain speed. Those who don't have the stomach to go through the math can skim the next few paragraphs.

We cannot derive equations for air resistance here. We'll just look them up. My source is *Fluid Mechanics*, by L. D. Landau and E. M. Lifshitz, two eminent Russian physicists. They give the following approximate formula:

$$F = \frac{1}{2} C_d A \rho v^2$$

The factors in this equation are the following:

$C_d =$ coefficient of friction, a factor depending on the shape of a car and determined by experiment; for a late model Corvette it is about 0.30;

$A =$ frontal area of the car; for a Corvette, it is about 20 square feet;

$\rho =$ Greek letter *rho*, density of air, which we calculate below;

$v =$ speed of the car.

Let us calculate the density of air using "back of the envelope" methods. We know that air is about 79% Nitrogen and 21% Oxygen. We can look up the fact that Nitrogen has a molecular weight of about 28 and Oxygen has a molecular weight of about 32. What is molecular weight? It is the mass (not the weight, despite the name) of 22.4 liters of gas. It is a number of historical convention, just like feet and inches, and doesn't have any real science behind it. So, we figure that air has an average molecular weight of

$$\frac{79\% \text{ of } 28 + 21\% \text{ of } 32 = 28.84 \text{ grams}}{22.4 \text{ liters}} = 1.29 \text{ gm/l}$$

I admit to using a calculator to do this calculation, against the spirit of the "back of the envelope" style. So sue me.

We need to convert 1.29 gm/l to pounds of mass per cubic foot so that we can do the force calculations in familiar, if not convenient, units. It is worthwhile to note, as an aside, that a great deal of the difficulty of doing calculations in the physics of racing has to do with the traditional units of feet, miles, and pounds we use. The metric system makes all such calculations vastly simpler. Napoleon Bonaparte wanted to convert the world the metric system (mostly so his own soldiers could do artillery calculations quickly in their heads) but it is still not in common use in America nearly 200 years later!

Again, we look up the conversion factors. My source is *Engineering Formulas* by Kurt Gieck, but they can be looked up in almost any encyclopedia or dictionary. There are 1000 liters in a cubic meter, which in turn contains 35.51 cubic feet. Also, a pound-mass contains 453.6 grams. These figures give us, for the density of air

$$1.29\,\frac{\text{gm}}{\text{liter}}\,\frac{\text{lb-mass}}{453.6\text{ gm}}\,\frac{1000\text{ liters}}{1\text{ meter}^3}\,\frac{1\text{ meter}^3}{35.51\text{ ft}^3} = 0.0801\,\frac{\text{lb-mass}}{\text{ft}^3}$$

This says that a cubic foot of air weighs 8 hundredths of a pound, and so it does! Air is much more massive than it seems, until you are moving quickly through it, that is.

Let's finish off our equation for air resistance. We want to fill in all the numbers except for speed, v, using the Corvette as an example car so that we can calculate the force of air resistance for a variety of speeds. We get

$$F = \frac{1}{2}\,(0.30 = C_d)\left(20\text{ ft}^2 = A\right)\left(0.080\,\frac{\text{lb-mass}}{\text{ft}^3} = \rho\right)v^2 = 0.24v^2\,\frac{\text{lb-mass}}{\text{ft}}$$

We want, at the end, to have v in miles per hour, but we need v in feet per seconds for the calculations to come out right. We recall that there are 22 feet per second for every 15 miles per hour, giving us

$$\begin{aligned} F &= 0.24\left(\frac{22\text{ ft/sec}}{15\text{ mph}}\,v\,(\text{mph})\right)^2\frac{\text{lb-mass}}{\text{ft}} \\ &= 0.517\,(v\ \ (\text{mph}))^2\,\frac{\text{lb-mass ft}}{\text{sec}^2} \end{aligned}$$

Now (this gets confusing, and it wouldn't be if we were using the metric system), a pound mass is a phony unit. A lb-mass is concocted to have a weight of 1 pound under the action of the Earth's gravity. Pounds are a unit of force or weight, not of mass. We want our force of air resistance in pounds of force, so we have to divide lb-mass ft/sec^2 by 32.1, numerically equal to the acceleration of Earth's gravity in ft/sec^2, to get pounds of force. You just have to know these things. This was a lot of work, but it's over now. We finally get

$$F = \frac{0.517}{32.1} \left(v \ \text{(mph)}\right)^2 = 0.0161 \left(v \ \text{(mph)}\right)^2 \text{ pounds}$$

Let's calculate a few numbers. The following table gives the force of air resistance for a number of interesting speeds:

v(mph)	15	30	60	90	120	150
F(pounds)	3.60	14.5	58.0	130	232	362

We can see that the force of air resistance goes up rapidly with speed, until we need over 350 pounds of constant force just to overcome drag at 150 miles per hour. We can now show where horsepower comes in.

Horsepower is a measure of *power*, which is a technical term in physics. It measures the amount of work that a force does as it acts over time. *Work* is another technical term in physics. It measures the actual effect of a force in moving an object over a distance. If we move an object one foot by applying a force of one pound, we are said to be doing one foot-pound of work. If it takes us one second to move the object, we have exerted one foot-pound per second of power. A horsepower is 550 foot-pounds per second. It is another one of those historical units that Napoleon hated and that has no reasonable origin in science.

We can expend one horsepower by exerting 550 pounds of force to move an object 1 foot in 1 second, or by exerting 1 pound of force to move an object 550 feet in 1 second, or by exerting 1 pound of force to move an object 1 foot in 0.001818 seconds, and so on. All these actions take the same amount of power. Incidentally, a horsepower happens to be equal also to 745 watts. So, if you burn about 8 light bulbs in your house, someone somewhere is expending at least one horsepower (and probably more like four or five) in electrical forces to keep all that going for you, and you pay for the service at the end of the month!.

All this means that to find out how much horsepower it takes to overcome air resistance at any speed, we need to multiply the force of air resistance by speed (in feet per second, converted from miles per hour), and divide by 550, to convert foot-lb/sec to horsepower. The formula is

$$P = Fv = \frac{0.0161}{550}\,\frac{22}{15}\,v^3 = \frac{0.354}{8{,}250}\,(v\ \text{(mph)})^3 \text{ horsepower}$$

and we get the following numbers from the formula for a few interesting speeds.

v(mph)	30	55	65	90	120	150	200
F(pounds)	14.5	48.7	68.0	130	232	362	644
horsepower	1.16	7.14	11.8	31.3	74.2	145	344

I put 55 mph and 65 mph in this table to show why some people think that the 55 mph national speed limit saves gasoline. It only requires about 7 hp to overcome drag at 55 mph, while it requires almost 12 hp to overcome drag at 65. Fuel consumption is approximately proportional to horsepower expended.

More interesting to the racer is the fact that it takes 145 hp to overcome drag at 150 mph. We know that our Corvette example car has about 240 hp, so about 95 hp must be going into overcoming rolling resistance and the slight braking forces arising from internal friction in the drive train and wheel bearings. Race cars capable of going 200 mph usually have at least 650 hp, about 350 of which goes into overcoming air resistance. It is probably possible to go 200 mph with a car in the 450–500 hp range, but such a car would have very good aerodynamics; expensive, low-friction internal parts; and low rolling resistance tires, which are designed to have the smallest possible contact patch like high performance bicycle tires, and are therefore not good for handling.

Part 7

The Traction Budget

This month, we introduce the traction budget. This is a way of thinking about the traction available for car control under various conditions. It can help you make decisions about driving style, the right line around a course, and diagnosing handling problems. We introduce a diagramming technique for visualizing the traction budget and combine this with a well-known visualization tool, the "circle of traction," also known as the circle of friction. So this month's article is about tools, conceptual and visual, for thinking about some aspects of the physics of racing.

To introduce the traction budget, we first need to visualize a tire in contact with the ground. Figure 1 shows how the bottom surface of a tire might look if we could see that surface by looking down from above. In other words, this figure shows an imaginary "X-ray" view of the bottom surface of a tire. For the rest of the discussion, we will always imagine that we view the tire this way. From this point of view, "up" on the diagram corresponds to forward forces and motion of the tire and the car, "down" corresponds to backward forces and motion, "left" corresponds to leftward forces and motion, and "right" on the diagram corresponds to rightward forces and motion.

The figure shows a shaded, elliptical region, where the tire presses against the ground. All the interaction between the tire and the ground takes place in this *contact patch*: that part of the tire that touches the ground. As the tire rolls, one bunch of tire molecules after another move into the contact patch. But the patch itself more-or-less keeps the same shape, size, and position relative to the axis of rotation of the tire and the car as a whole. We can use this fact to develop a simplified view of the interaction between tire and ground. This simplified view lets us quickly and easily do approximate calculations good within a few percent. (A full-blown, mathematical analysis requires tire coordinates that roll with the tire, ground coordinates fixed on the ground, car coordinates fixed to the car, and many complicated equations relating these coordinate systems; the last few percent of accuracy in a mathematical model of tire-ground interaction involves a great deal more complexity.)

You will recall that forces on the tire from the ground are required to make

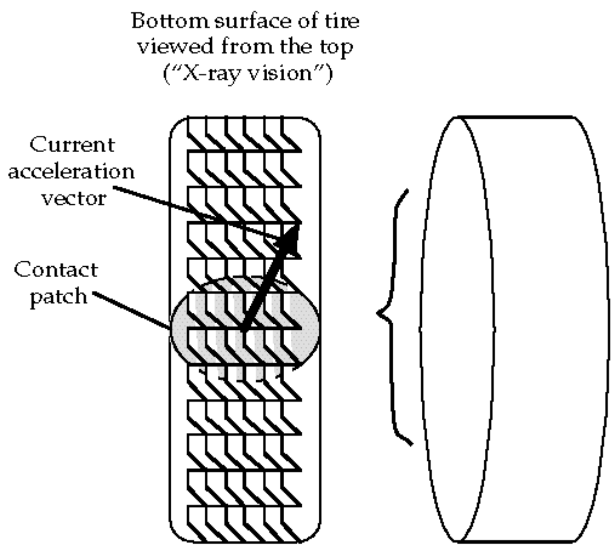


Figure 1: The bottom surface of a tire viewed from the top as though with "X ray vision."

a car change either its speed of motion or its direction of motion. Thinking of the X-ray vision picture, forces pointing up are required to make the car accelerate, forces pointing down are required to make it brake, and forces pointing right and left are required to make the car turn. Consider forward acceleration, for a moment. The engine applies a torque to the axle. This torque becomes a force, pointing backwards (down, on the diagram), that the tire applies to the ground. By Newton's third law, the ground applies an equal and opposite force, therefore pointing forward (up), on the contact patch. This force is transmitted back to the car, accelerating it forward. It is easy to get confused with all this backward and forward action and reaction. Remember to think only about the forces on the tire and to ignore the forces on the ground, which point the opposite way.

You will also recall that a tire has a limited ability to stick to the ground. Apply a force that is too large, and the tire slides. The maximum force that a tire can take depends on the weight applied to the tire:

$$F \leq \mu W$$

where F is the force on the tire, μ is the coefficient of adhesion (and depends on tire compound, ground characteristics, temperature, humidity, phase of the moon, *etc.*), and W is the weight or load on the tire.

By Newton's second law, the weight on the tire depends on the fraction of the car's mass that the tire must support and the acceleration of gravity, $g = 32.1\ \mathrm{ft/sec}^2$. The fraction of the car's mass that the tire must support depends on geometrical factors such as the wheelbase and the height of the center of gravity. It also depends on the acceleration of the car, which completely accounts for weight transfer.

It is critical to separate the geometrical, or *kinematic*, aspects of weight transfer from the mass of the car. Imagine two cars with the same geometry but different masses (weights). In a one g braking maneuver, the same *fraction* of each car's total weight will be transferred to the front. In the example of Part 1 of this series, we calculated a 20% weight transfer during one g braking because the height of the CG was 20% of the wheelbase. This weight transfer will be the same 20% in a 3500 pound, stock Corvette as in a 2200 pound, tube-frame, Trans-Am Corvette so long as the geometry (wheelbase, CG height, *etc.*) of the two cars is the same. Although the actual weight, in pounds, will be different in the two cases, the fractions of the cars' total weight will be equal.

Separating kinematics from mass, then, we have for the weight

$$W = f(a)mg$$

where $f(a)$ is the fraction of the car's mass the tire must support and also accounts for weight transfer, m is the car's mass, and g is the acceleration of gravity.

Finally, by Newton's second law again, the acceleration of the tire due to the force F applied to it is

$$a = F/f(a)m$$

We can now combine the expressions above to discover a fascinating fact:

$$a = F/f(a)m \leq a_{max}$$

$$a_{max} = \frac{\mu W}{f(a)m} = \frac{\mu f(a)mg}{f(a)m} = \mu g$$

The maximum acceleration a tire can take is μg, a constant, independent of the mass of the car! While the maximum *force* a tire can take depends very much on the current vertical load or weight on the tire, the acceleration of that tire does not depend on the current weight. If a tire can take one g before sliding, it can take it on a lightweight car as well as on a heavy car, and it can take it under load as well as when lightly loaded. We hinted at this fact in Part 2, but the analysis above hopefully gives some deeper insight into it. We note that a_{max} being constant is only approximately true, because μ changes slightly as tire load varies, but this is a second-order effect (covered in a later article).

So, in an approximate way, we can consider the available acceleration from a tire independently of details of weight transfer. The tire will give you so many gees and that's that. This is the essential idea of the traction budget. What you do with your budget is your affair. If you have a tire that will give you one g, you can use it for accelerating, braking, cornering, or some combination, but you cannot use more than your budget or you will slide. The front-back component of the budget measures accelerating and braking, and the right-left component measures cornering acceleration. The front-back component, call it a_y, combines with the left-right component, a_x, not by adding, but by the Pythagorean formula:

$$a = \sqrt{a_x^2 + a_y^2}$$

Rather than trying to deal with this formula, there is a convenient, visual representation of the traction budget in the *circle of traction.* Figure 2 shows the circle. It is oriented in the same way as the X-ray view of the contact patch, Figure 1 , so that up is forward and right is rightward. The circular boundary represents the limits of the traction budget, and every point inside the circle represents a particular choice of how you spend your budget. A point near the top of the circle represents pure, forward acceleration, a point near the bottom represents pure braking. A point near the right boundary, with no up or down component, represents pure rightward cornering acceleration. Other points represent Pythagorean combinations of cornering and forward or backward acceleration.

The beauty of this representation is that the effects of weight transfer are factored out. So the circle remains approximately the same no matter what the load on a tire.

In racing, of course, we try to spend our budget so as to stay as close to the limit, *i.e.* , the circular boundary, as possible. In street driving, we try to stay well inside the limit so that we have lots of traction available to react to unforeseen circumstances.

I have emphasized that the circle is only an approximate representation of the truth. It is probably close enough to make a computer driving simulation

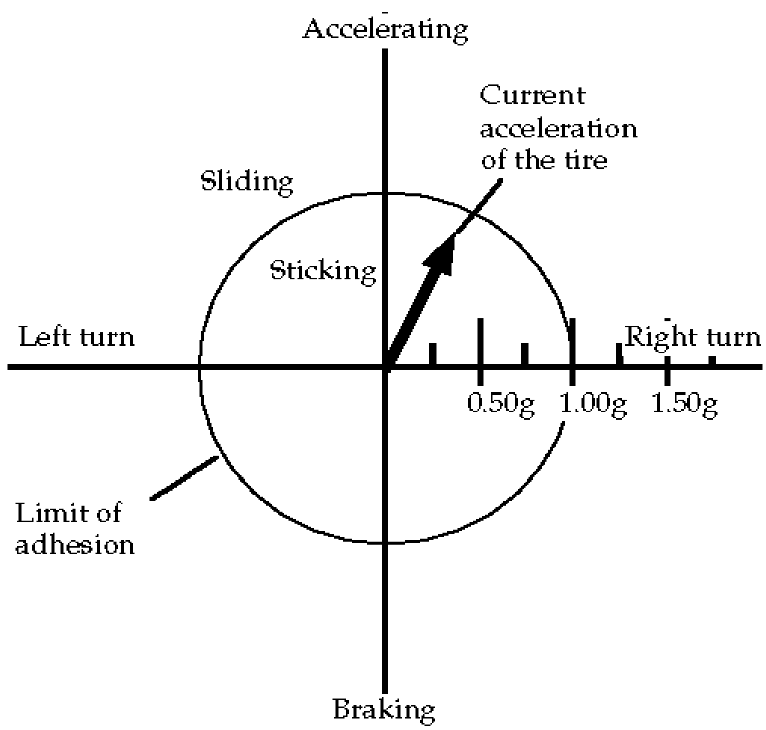


Figure 2: The Circle of Traction

that feels right (I'm pretty sure that "Hard Drivin' " and other such games use it). As mentioned, tire loads do cause slight, dynamic variations. Car characteristics also give rise to variations. Imagine a car with slippery tires in the back and sticky tires in the front. Such a car will tend to oversteer by sliding. Its traction budget will not look like a circle. Figure 3 gives an indication of what the traction budget for the whole car might look like (we have been discussing the budget of a single tire up to this point, but the same notions apply to the whole car). In Figure 3 , there is a large traction circle for the sticky front tires and a small circle for the slippery rear tires. Under acceleration, the slippery rears dominate the combined traction budget because of weight transfer. Under braking, the sticky fronts dominate. The combined traction budget looks something like an egg, flattened at top and wide in the middle. Under braking, the traction available for cornering is considerably greater than the traction available during acceleration because the sticky fronts are working. So, although this poorly handling car tends to oversteer by sliding the rear, it also tends to understeer during acceleration because the slippery rears will not follow the steering front tires very effectively.

The traction budget is a versatile and simple technique for analyzing and visualizing car handling. The same technique can be applied to developing driver's skills, planning the line around a course, and diagnosing handling problems.

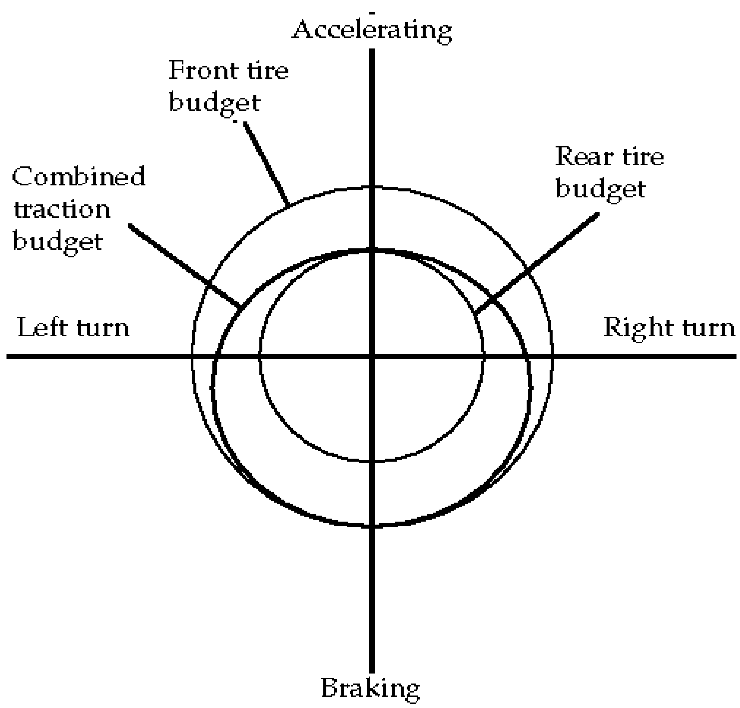


Figure 3: A traction budget diagram for a poorly handling car.

Part 8

Simulating Car Dynamics with a Computer Program

This month, we begin writing a computer program to simulate the physics of racing. Such a program is quite an ambitious one. A simple racing video game, such as "Pole Position," probably took an expert programmer several months to write. A big, realistic game like "Hard Drivin'" probably took three to five people more than a year to create. The point is that the topic of writing a racing simulation is one that we will have to revisit many times in these articles, assuming your patience holds out. There are many 'just physics' topics still to cover too, such as springs and dampers, transients, and thermodynamics. Your author hopes you will find the computer programming topic an enjoyable sideline and is interested, as always, in your feedback.

We will use a computer programming language called Scheme. You have probably encountered BASIC, a language that is very common on personal computers. Scheme is like BASIC in that it is *interactive*. An interactive computer language is the right kind to use when inventing a program as you go along. Scheme is better than BASIC, however, because it is a good deal simpler and also more powerful and modern. Scheme is available for most PCs at very modest cost (MIT Press has published a book and diskette with Scheme for IBM compatibles for about $40; I have a free version for Macintoshes). I will explain everything we need to know about Scheme as we go along. Although I assume little or no knowledge about computer programming on your part, we will ultimately learn some very advanced things.

The first thing we need to do is create a *data structure* that contains the mathematical state of the car at any time. This data structure is a block of computer memory. As simulated time progresses, mathematical operations performed on the data structure simulate the physics. We create a new instance of this data structure by typing the following on the computer keyboard at the Scheme prompt:

```
(new-race-car)
```

This is an example of an *expression*. The expression includes the parentheses. When it is typed in, it is `evaluated` immediately. When we say that Scheme is an interactive programming language, we mean that it evaluates expressions immediately. Later on, I show how we *define* this expression. It is by defining such expressions that we write our simulation program.

Everything in Scheme is an expression (that's why Scheme is simple). Every expression has a value. The value of the expression above is the new data structure itself. We need to give the new data structure a name so we can refer to it in later expressions:

```
(define car-161 (new-race-car))
```

This expression illustrates two Scheme features. The first is that expressions can contain sub-expressions inside them. The inside expressions are called *nested*. Scheme figures out which expressions are nested by counting parentheses. It is partly by nesting expressions that we build up the complexity needed to simulate racing. The second feature is the use of the special Scheme word `define`. This causes the immediately following word to become a stand-in synonym for the value just after. The technical name for such a stand-in synonym is *variable*. Thus, the expression `car-161`, wherever it appears after the `define` expression, is a synonym for the data structure created by the nested expression `(new-race-car)`.

We will have another data structure (with the same format) for `car-240`, another for `car-70`, and so on. We get to choose these names to be almost anything we like [1]. So, we would create all the data structures for the cars in our simulation with expressions like the following:

```
(define car-161 (new-race-car))
(define car-240 (new-race-car))
(define car-70  (new-race-car))
```

The state of a race car consists of several numbers describing the physics of the car. First, there is the car's position. Imagine a map of the course. Every position on the map is denoted by a pair of coordinates, x and y. For elevation changes, we add a height coordinate, z. The position of the center of gravity of a car at any time is denoted with expressions such as the following:

[1] It so happens, annoyingly, that we can't use the word `car`. This is a Scheme reserved word, like `define`. Its use is explained later

```
(race-car-x car-161)
(race-car-y car-161)
(race-car-z car-161)
```

Each of these expressions performs *data retrieval* on the data structure `car-161`. The value of the first expression is the x coordinate of the car, *etc.*Normally, when running the Scheme interpreter, typing an expression simply causes its value to be printed, so we would see the car position coordinates printed out as we typed. We could also store these positions in another block of computer memory for further manipulations, or we could specify various mathematical operations to be performed on them.

The next pieces of state information are the three components of the car's velocity. When the car is going in any direction on the course, we can ask "how fast is it going in the x direction, ignoring its motion in the y and z directions?" Similarly, we want to know how fast it is going in the y direction, ignoring the x and z directions, and so on. Decomposing an object's velocity into separate components along the principal coordinate directions is necessary for computation. The technique was originated by the French mathematician Descartes, and Newton found that the motion in each direction can be analyzed independently of the motions in the other directions at right angles to the first direction.

The velocity of our race car is retrieved via the following expressions:

```
(race-car-vx car-161)
(race-car-vy car-161)
(race-car-vz car-161)
```

To end this month's article, we show how velocity is computed. Suppose we retrieve the position of the car at simulated time t_1 and save it in some variables, as follows:

```
(define x1 (race-car-x car-161))
(define y1 (race-car-y car-161))
(define z1 (race-car-z car-161))
```

and again, at a slightly later instant of simulated time, t_2:

```
(define x2 (race-car-x car-161))
(define y2 (race-car-y car-161))
(define z2 (race-car-z car-161))
```

We have used `define` to create some new variables that now have the values of the car's positions at two times. To calculate the average velocity of the car between the two times and store it in some more variables, we evaluate the following expressions:

```
(define vx (/ (- x2 x1) (- t2 t1)))
(define vy (/ (- y2 y1) (- t2 t1)))
(define vz (/ (- z2 z1) (- t2 t1)))
```

The nesting of expressions is one level deeper than we have seen heretofore, but these expressions can be easily analyzed. Since they all have the same form, it suffices to explain just one of them. First of all, the `define` operation works as before, just creating the variable `vx` and assigning it the value of the following expression. This expression is

```
(/ (- x2 x1) (- t2 t1))
```

In normal mathematical notation, this expression would read

$$\frac{x_2 - x_1}{t_2 - t_1}$$

and in most computer languages, it would look like this:

```
(x2 - x1) / (t2 - t1)
```

We can immediately see this is the velocity in the x direction: a change in position divided by the corresponding change in time. The Scheme version of this expression looks a little strange, but there is a good reason for it: consistency. Scheme requires that all operations, including everyday mathematical ones, appear in the first position in a parenthesized expression, immediately after the left parenthesis. Although consistency makes mathematical expressions look strange, the payback is simplicity: all expressions have the same form. If Scheme had one notation for mathematical expressions and another notation for non-mathematical expressions, like most computer languages, it would be more complicated. Incidentally, Scheme's notation is called Polish notation. Perhaps you have been exposed to Hewlett-Packard calculators, which use reverse Polish, in in which the operator always appears in the *last* position. Same idea, and advantages, as Scheme, only reversed.

So, to analyze the expression completely, it is a division expression

```
(/ ...)
```

whose two arguments are nested subtraction expressions

```
(- ...) (- ...)
```

The whole expression has the form

```
(/ (- ...) (- ...))
```

which, when the variables are filled in, is

```
(/ (- x2 x1) (- t2 t1))
```

After a little practice, Scheme's style for mathematics becomes second nature and the advantages of consistent notation pay off in the long run.

Finally, we should like to store the velocity values in our data structure. We do so as follows:

```
(set-race-car-vx! car-161 vx)
(set-race-car-vy! car-161 vy)
(set-race-car-vz! car-161 vz)
```

The `set` operations change the values in the data structure named `car-161`. The exclamation point at the end of the names of these operations doesn't do anything special. It's just a Scheme idiom for operations that change data structures.

Part 9

Straights

We found in part 5 of this series, "Introduction to the Racing Line," that a driver can lose a shocking amount of time by taking a bad line in a corner. With a six-foot-wide car on a ten-foot-wide course, one can lose sixteen hundredths by 'blowing' a single right-angle turn. This month, we extend the analysis of the racing line by following our example car down a straight. It is often said that the most critical corner in a course is the one before the longest straight. Let's find out how critical it is. We calculate how much time it takes to go down a straight as a function of the speed entering the straight. The results, which are given at the end, are not terribly dramatic, but we make several, key improvements in the mathematical model that is under continuing development in this series of articles. These improvements will be used as we proceed designing the computer program begun in Part 8.

The mathematical model for traveling down a straight follows from Newton's second law:

$$F = ma, \tag{1}$$

where F is the force on the car, m is the mass of the car, and a is the acceleration of the car. We want to solve this equation to get time as a function of distance down the straight. Basically, we want a table of numbers so that we can look up the time it takes to go any distance. We can build this table using accountants' columnar paper, or we can use the modern version of the columnar pad: the electronic spreadsheet program.

To solve equation 1, we first invert it:

$$a = F/m. \tag{2}$$

Now a, the acceleration, is the rate of change of velocity with time. *Rate of change* is simply the ratio of a small change in velocity to a small change in time. Let us assume that we have filled in a column of times on our table. The times start with 0 and go up by the same, small amount, say 0.05 sec. Physicists call this small time the *integration step*. It is standard practice to begin solving an equation with a fixed integration step. There are sometimes good reasons to vary the integration step, but those reasons do not arise in this problem. Let us call the integration step Δt. If we call the time in the

i-th row t_i, then for every row except the first,

$$\Delta t = t_i - t_{i-1} = \text{constant}. \tag{3}$$

We label another column *velocity*, and we'll call the velocity in the i-th row v_i. For every row except the first, equation 2 becomes:

$$\frac{v_i - v_{i-1}}{\Delta t} = F/m. \tag{4}$$

We want to fill in velocities as we go down the columns, so we need to solve equation 4 for v_i. This will give us a formula for computing v_i given v_{i-1} for every row except the first. In the first row, we put the speed with which we enter the straight, which is an input to the problem. We get:

$$v_i = v_{i-1} + \Delta t F/m. \tag{5}$$

We label another column *distance*, and we call the distance value in the i-th row x_i. Just as acceleration is the rate of change of velocity, so velocity is the rate of change of distance over time. Just as before, then, we may write:

$$v_i = \frac{x_i - x_{i-1}}{\Delta t}. \tag{6}$$

Solved for x_i, this is:

$$x_i = x_{i-1} + \Delta t v_i. \tag{7}$$

Equation 7 gives us a formula for calculating the distance for any time given the previous distance and the velocity calculated by equation 5. Physicists would say that we have a scheme for *integrating the equations of motion*.

A small detail is missing: what is the force, F? Everything to this point is *kinematic*. The real modeling starts now with formulas for calculating the force. For this, we will draw on all the previous articles in this series. Let's label another column *force*, and a few more with *drag*, *rolling resistance*, *engine torque*, *engine rpm*, *wheel rpm*, *trans gear ratio*, *drive ratio*, *wheel torque*, and *drive force*. As you can see, we are going to derive a fairly complete, if not accurate, model of accelerating down the straight. We need a few constants:

CONSTANT	SYMBOL	EXAMPLE VALUE
rear end ratio	R	3.07
density of air	ρ	0.0025 slugs/ft^3
coeff. of drag	C_d	0.30
frontal area	A	20 ft^2
wheel diameter	d	26 in = 2.167 ft
roll resist factor	r_r	0.696 lb/(ft/sec)
car mass	m	100 slug
first gear ratio	g_1	2.88
second gear ratio	g_2	1.91
third gear ratio	g_3	1.33
fourth gear ratio	g_4	1.00

and a few variables:

VARIABLE	SYMBOL	EXAMPLE VALUE
engine torque	T_E	330 ft-lbs
drag	F_d	45 lbs
rolling resistance	F_r	54 lbs
engine rpm	E	4000
wheel rpm	W	680
wheel torque	T_W	1930 ft-lbs
wheel force	F_W	1780 lbs
net force	F	1681 lbs

All the example values are for a late model Corvette. *Slugs* are the English unit of mass, and 1 slug weighs about 32.1 lbs at sea level (another manifestation of $F = ma$, with F in lbs, m in slugs, and a being the acceleration of gravity, 32.1 ft/sec$_2$).

The most basic modeling equation is that the force we can use for forward acceleration is the propelling force transmitted through the wheels minus drag and rolling resistance:

$$F = F_W - F_d - F_r. \tag{8}$$

The force of drag we get from Part 6:

$$F_d = \frac{1}{2} C_d A \rho v_i^2. \tag{9}$$

Note that to calculate the force at step i, we can use the velocity at step i. This force goes into calculating the acceleration at step i, which is used to calculate the velocity and distance at step $i+1$ by equations 5 and 7. Those two equations represent the only 'backward references' we need. Thus, the only inputs to the integration are the initial distance, 0, and the entrance velocity, v_0.

The rolling resistance is approximately proportional to the velocity:

$$F_r = r_r v_i = 0.696 v_1. \tag{10}$$

This approximation is probably the weakest one in the model. I derived it by noting from a Corvette book that 8.2 hp were needed to overcome rolling resistance at 55 mph. I have nothing else but intuition to go on for this equation, so take it with a grain of salt.

Finally, we must calculate the forward force delivered by the ground to the car by reaction to the rearward force delivered to the ground *via* the engine and drive train:

$$F_W = \frac{T_E R g_k}{d/2}. \tag{11}$$

This equation simply states that we take the engine torque multiplied by the rear axle ratio and the transmission drive ratio in the k-th gear, which is the torque at the drive wheels, T_W, and divide it by the radius of the wheel, which is half the diameter of the wheel, d.

To calculate the forward force, we must decide what gear to be in. The logic we use to do this is the following: from the velocity, we can calculate the wheel rpm:

$$W = 60\,\frac{\text{sec}}{\text{min}}\,\frac{v_i}{\pi d}. \tag{12}$$

From this, we know the engine rpm:

$$E = W R g_k. \tag{13}$$

At each step of integration, we look at the current engine rpm and ask "is it past the torque peak of the engine?" If so, we shift to the next highest gear, if possible. Somewhat arbitrarily, we assume that the torque peak is at 4200 rpm. To keep things simple, we also make the optimistic assumption that the engine puts out a constant torque of 330 ft-lbs. To make the model more realistic, we need merely look up a torque curve for our engine, usually expressed as a function of rpm, and read the torque off the curve at each step of the integration. The current approximation is not terrible however; it merely gives us artificially good times and speeds. Another important

Table 1: Exit speeds and times for several entrance speeds

	200 ft straight		500 ft straight	
Entrance speed (mph)	Exit speed (mph)	Time (sec)	Exit speed (mph)	Time (sec)
25	61.51	2.972	81.12	5.811
27	61.77	2.916	81.51	5.748
29	62.15	2.845	82.02	5.676
31	62.34	2.793	82.19	5.599
35	63.18	2.691	82.78	5.472
40	64.65	2.548	83.49	5.282
45	66.85	2.392	84.68	5.065
50	69.27	2.261	85.83	4.875

improvement on the logic would be to check whether the wheels are spinning, *i.e.* , that acceleration is less than about $\frac{1}{2}G$, and to 'lift off the gas' in that case.

We have all the ingredients necessary to calculate how much time it takes to cover a straight given an initial speed. You can imagine doing the calculations outlined above by hand on columnar paper, or you can check my results (below) by programming them up in a spreadsheet program like Lotus 1-2-3 or Microsoft Excel. Eventually, of course, if you follow this series, you will see these equations again as we write our Scheme program for simulating car dynamics. Integrating the equations of motion by hand will take you many hours. Using a spreadsheet will take several hours, too, but many less than integrating by hand.

To illustrate the process, we show below the times and exit speeds for a 200 foot straight, which is a fairly long one in autocrossing, and a 500 foot straight, which you should only see on race tracks. We show times and speeds for a variety of speeds entering the straight from 25 to 50 mph in Table 1. The results are also summarized in the two plots, Figures 1 and 2.

The notable facts arising in this analysis are the following. The time difference resulting from entering the 200' straight at 27 mph rather than 25 mph is about 6 hundredths. Frankly, not as much as I expected. The

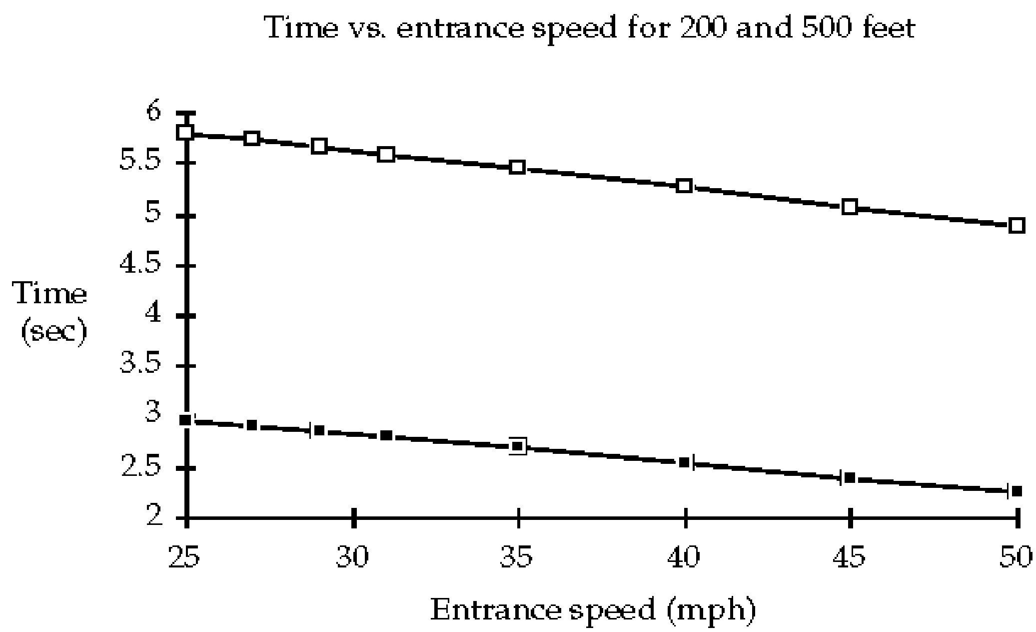

Time vs. entrance speed for 200 and 500 feet
Time (sec)
6
5.5
5
4.5
4
3.5
3
2.5
2
25
30
35
40
45
50
Entrance speed (mph)

Figure 1:

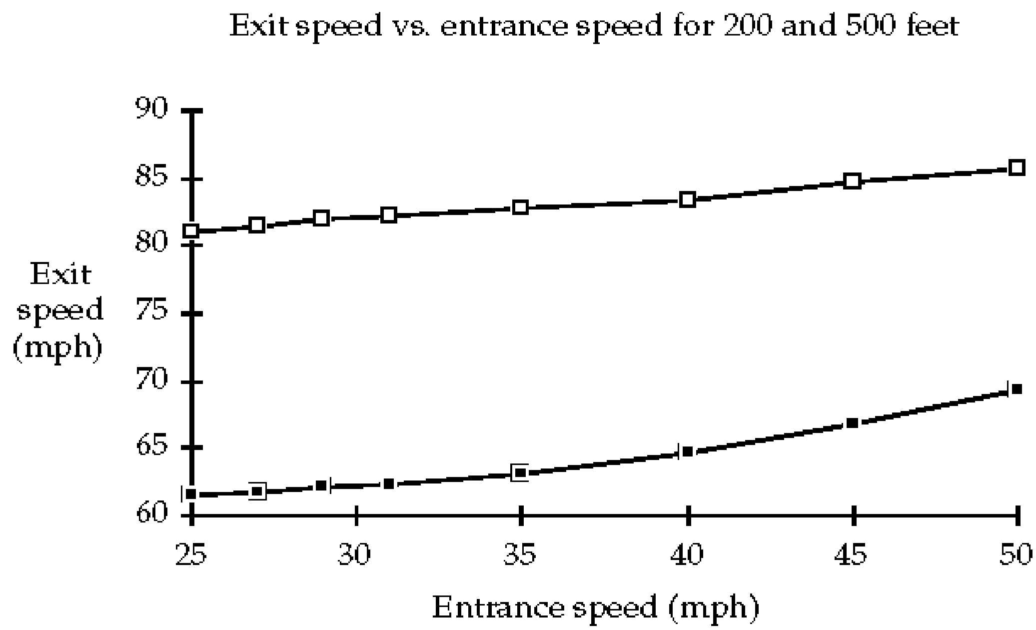

Exit speed vs. entrance speed for 200 and 500 feet
90
85
80
75
70
65
60
Exit
speed
(mph)
25
30
35
40
45
50
Entrance speed (mph)

Figure 2:

time difference between entering at 31 mph over 25 mph is about 2 tenths, again less than I would have guessed. The speed difference at the end of the straight between entering at 25 mph and 50 mph is only 8 mph, a result of the fact that the car labors against friction and higher gear ratios at high speeds. It is also a consequence of the fact that there is so much torque available at 25 mph in low gear that the car can almost make up the difference over the relatively short 200' straight. In fact, on the longer 500' straight, the exit speed difference between entering at 25 mph and 50 mph is not even 5 mph, though the time difference is nearly a full second.

This analysis would most likely be much more dramatic for a car with less torque than a Corvette. In a Corvette, with 330 ft-lbs of torque on tap, the penalty for entering a straight slower than necessary is not so great as it would be in a more typical car, where recovering speed lost through timidity or bad cornering is much more difficult.

Again, the analysis can be improved by using a real torque curve and by checking whether the wheels are spinning in lower gears.

Part 10
Grip Angle

In many ways, tire mechanics is an unpleasant topic. It is shrouded in uncertainty, controversy, and trade secrecy. Both theoretical and experimental studies are extremely difficult and expensive. It is probably the most uncontrollable variable in racing today. As such, it is the source of many highs and lows. An improvement in modeling or design, even if it is found by lucky accident, can lead to several years of domination by one tire company, as with BFGoodrich in autocrossing now. An unfortunate choice of tire by a competitor can lead to frustration and a disastrous hole in the budget.

This month, we investigate the physics of tire adhesion a little more deeply than in the past. In Parts 2, 4, and 7, we used the simple friction model given by $F \leq \mu W$, where F is the maximum traction force available from a tire; μ, assumed constant, is the coefficient of friction; and W is the instantaneous vertical load, or weight, on a tire. While this model is adequate for a rough, intuitive feel for tire behavior, it is grossly inadequate for quantitative use, say, for the computer program we began in Part 8 or for race car engineering and set up.

I am not a tire engineer. As always, I try to give a fresh look at any topic from a physicist's point of view. I may write things that are heretical or even wrong, especially on such a difficult topic as tire mechanics. I invite debate and corrections from those more knowledgeable than I. Such interaction is part of the fun of these articles for me.

I call this month's topic 'grip angle.' The grip angle is a quantity that captures, for many purposes, the complex and subtle mechanics of a tire. Most writers call this quantity 'slip angle.' I think this name is misleading because it suggests that a tire works by slipping and sliding. The truth is more complicated. Near maximum loads, the contact patch is partly gripping and partly slipping. The maximum net force a tire can yield occurs at the threshold where the tire is still gripping but is just about to give way to total slipping. Also, I have some difficulties with the analyses of slip angle in the literature. I will present these difficulties in these articles, unfortunately, probably without resolution. For these reasons, I give the quantity a new name.

A tire is an elastic or deformable body. It delivers forces to the car by stretching, compressing, and twisting. It is thus a very complex sort of spring with several different ways, or *modes*, of deformation. The hypothetical tire implied by $F \leq \mu W$ with constant μ would be a non-elastic tire. Anyone who has driven hard tires on ice knows that non-elastic tires are basically uncontrollable, not just because μ is small but because regular tires on ice do not twist appreciably.

The first and most obvious mode of derformation is radial. This deformation is along the radius of the tire, the line from the center to the tread. It is easily visible as a bulge in the sidewall near the contact patch, where the tire touches the ground. Thus, radial compression varies around the circumference.

Second is circumferential deformation. This is most easily visble as wrinkling of the sidewalls of drag tires. These tires are intentionally set up to deform dramatically in the circumferential direction.

Third is axial deformation. This is a deflection that tends to pull the tire off the (non-elastic) wheel or rim.

Last, and most important for cornering, is *torsional* deformation. This is a difference in axial deflection from the front to the back of the contact patch. Fundamentally, radial, circumferential, and axial deformation furnish a complete description of a tire. But it is very useful to consider the *differences* in these deflections around the circumference.

Let us examine exactly how a tire delivers cornering force to the car. We can get a good intuition into the physics with a pencil eraser. Get a block eraser, of the rectangular kind like 'Pink Pearl' or 'Magic Rub.' Stand it up on a table or desk and think of it as a little segment of the circumference of a tire. Think of the part touching the desk as the contact patch. Grab the top of the eraser and think of your hand as the wheel or rim, which is going to push, pull, and twist on the segment of tire circumference as we go along the following analysis.

Consider a car traveling at speed v in a straight line. Let us turn the steering wheel slightly to the right (twist the top of the eraser to the right). At the instant we begin turning, the rim (your hand on the eraser), at a circumferential position just behind the contact patch, pushes slightly leftward on the bead of the tire. Just ahead of the contact patch, likewise, the rim pulls the bead a little to the right. The push and pull together are called a *force couple*. This couple delivers a torsional, clockwise stress to the inner part of the tire carcass, near the bead. This stress is communicated to the contact patch by the elastic material in the sidewalls (or the main body of the eraser). As a result of turning the steering wheel, therefore, the rim twists the contact patch clockwise.

The car is still going straight, just for an instant. How are we going to explain a net rightward force from the road on the contact patch? This net force *must* be there, otherwise the tire and the car would continue in a straight line by Newton's First Law.

Consider the piece of road just under the contact patch at the instant the turn begins. The rubber particles on the left side of the patch are going a little bit faster with respect to the road than the rest of the car and the rubber particles on the right side of the patch are going a little bit slower than the rest of the car. As a result, the left side of the patch grips a little bit less than the right. The rubber particles on the left are more likely to slide and the ones on the right are more likely to grip. Thus, the left edge of the patch 'walks' a little bit upward, resulting in a net clockwise twisting motion of the patch. The torsional stress becomes a torsional motion. As this motion is repeated from one instant to the next, the tire (and the eraser—I hope you are still following along with the eraser) walks continuosly to the right.

The better grip on the right hand side of the contact patch adds up to a net rightward force on the tire, which is transmitted back through the sidewall to the car. The chassis of the car begins to yaw to the right, changing the direction of the rear wheels. A torsional stress on the rear contact patches results, and the rear tires commence a similar 'walking' motion.

The wheel (your hand) is twisted more away from the direction of the car than is the contact patch. The angular difference between the direction the wheel is pointed and the direction the tire walks is the grip angle. All quantities of interest in tire mechanics—forces, friction coefficients, *etc.*, are conventionally expressed as functions of grip angle.

In steady state cornering, as in sweepers, an understeering car has larger grip angles in front, and an oversteering car has larger grip angles in the rear. How to control grip angles statically with wheel alignment and dynamically with four-wheel steering are subjects for later treatment.

The greater the grip angle, the larger the cornering force becomes, up to a point. After this point, greater grip angle delivers less force. This point is analogous to the idealized adhesive limit mentioned earlier in this series. Thus, a real tire behaves *qualitatively* like an ideal tire, which grips until the adhesive limit is exceeded and then slides. A real tire, however, grips gradually better as cornering force increases, and then grips gradually worse as the limit is exceeded.

The walking motion of the contact patch is not entirely smooth, or in otherwords, somewhat *discrete*. Individual blocks of rubber alternately grip and slide at high frequency, thousands of times per second. Under hard cornering, the rubber blocks vibrating on the road make an audible squaling sound. Beyond the adhesive limit, squealing becomes a lower frequency sound, 'squalling,' as the point of optimum efficiency of the walking process is bypassed.

There is a lot more to say on this subject, and I admit that my first attempts at a mathematical analysis of grip angle and contact patch mechanics got bogged down. However, I think we now have an intuitive, conceptual basis for better modeling in the future.

Speaking of the future, summarize briefly the past of and plans for the *Physics of Racing* series. The following overlapping threads run through it:

Tire Physics concerns adhesion, grip angle, and elastic modeling. This has been covered in Parts 2, 4, 7, and 10, and will be covered in several later parts.

Car Dynamics concerns handling, suspension movement, and motion of a car around a course; has been covered in Parts 1, 4, 5, and 8 and will continue.

Drive Line Physics concerns modeling of engine performance and acceleration. Has been covered in Parts 3, 6, and 9 and will also continue.

Computer Simulation concerns the design of a working program that captures all the physics. This is the ultimate goal of the series. It was begun in Part 8 and will eventually dominate discussion.

The following is a list of articles that have appeared so far:

1. Weight Transfer

2. Keeping Your Tires Stuck to the Ground

3. Basic Calculations

4. There is No Such Thing as Centrifugal Force

5. Introduction to the Racing Line

6. Speed and Horsepower

7. The Circle of Traction

8. Simulating Car Dynamics with a Computer Program

9. Straights

10. Grip Angle

and the following is a *tentative* list of articles I have planned for the near future (naturally, this list is 'subject to change without notice'):

Springs and Dampers, presenting a detailed model of suspension movement (suggested by Bob Mosso)

Transients, presenting the dynamics of entering and leaving corners, chicanes, and slaloms (this one suggested by Karen Babb)

Stability, explaining why spins and other losses of control occur

Smoothness, exploring what, exactly, is meant by smoothness

Modeling Car Data in a computer program; in several articles

Modeling Course Data in a computer program; also in several articles

In practice, I try to keep the lengths of articles about the same, so if a topic is getting too long (and grip angle definitely did), I break it up in to several articles.

Part 11
Braking

I was recently helping to crew Mark Thornton's effort at the Silver State Grand Prix in Nevada. Mark had built a beautiful car with a theoretical top speed of over 200 miles per hour for the 92 mile time trial from Lund to Hiko. Mark had no experience driving at these speeds and asked me as a physicist if I could predict what braking at 200 mph would be like. This month I report on the back-of-the-envelope calculations on braking I did there in the field.

There are a couple of ways of looking at this problem. Brakes work by converting the energy of motion, *kinetic* energy, into the energy of heat in the brakes. Converting energy from useful forms (motion, electrical, chemical, *etc.*) to heat is generally called *dissipating* the energy, because there is no easy way to get it back from heat. If we assume that brakes dissipate energy at a constant rate, then we can immediately conclude that it takes four times as much time to stop from 200 mph as from 100 mph. The reason is that kinetic energy goes up as the square of the speed. Going at twice the speed means you have four times the kinetic energy because $4 = 2^2$. The exact formula for kinetic energy is $\frac{1}{2}mv^2$, where m is the mass of an object and v is its speed. This was useful to Mark because braking from 100 mph was within the range of familiar driving experience.

That's pretty simple, but is it right? Do brakes dissipate energy at a constant rate? My guess as a physicist is 'probably not.' The efficiency of the braking process, dissipation, will depend on details of the friction interaction between the brake pads and disks. That interaction is likely to vary with temperature. Most brake pads are formulated to grip harder when hot, but only up to a point. Brake fade occurs when the pads and rotors are overheated. If you continue braking, heating the system even more, the brake fluid will eventually boil and there will be no braking at all. Brake fluid has the function of transmitting the pressure of your foot on the pedal to the brake pads by hydrostatics. If the fluid boils, then the pressure of your foot on the pedal goes into crushing little bubbles of gaseous brake fluid in the brake lines rather than into crushing the pads against the disks. Hence, no brakes.

We now arrive at the second way of looking at this problem. Let us assume

Starting Speed (mph)	Starting Speed (fps)	Time to brake (sec)	Distance to brake (feet)	Distance to brake (yards)
30	44	1.37	30.16	10.05
60	88	2.74	120.62	40.21
90	132	4.11	271.40	90.47
120	176	5.48	482.49	160.83
150	220	6.85	753.89	251.30
180	264	8.22	1085.61	361.87
210	308	9.60	1477.63	492.54

Table 2: Times and distances for braking to zero from various speeds.

that we have good brakes, so that the braking process is limited *not* by the interaction between the pads and disks but by the interaction between the tires and the ground. In other words, let us assume that our brakes are better than our tires. To keep things simple and back-of-the-envelope, assume that our tires will give us a constant deceleration of

$$1G \equiv a = 32.1 \frac{\text{feet}}{\text{sec}^2}$$

The time t required for braking from speed v can be calculated from:

$$t = v/a$$

which simply follows from the definition of constant acceleration. Given the time for braking, we can calculate the distance x, again from the definitions of acceleration and velocity:

$$x = vt - \frac{1}{2}at^2$$

Remembering to be careful about converting miles per hour to feet per second, we arrive at the numbers in Table 1.

We can immediately see from this table (and, indeed, from the formulas) that it is the *distance*, not the time, that varies as the square of the starting speed v. The braking time only goes up linearly with speed, that is, in simple proportion.

The numbers in the table are in the ballpark of the braking figures one reads in published tests of high performance cars, so I am inclined to believe that the second way of looking at the problem is the right way. In other words, the assumption that the brakes are better than the tires, so long as they are not overheated, is probably right, and the assumption that brakes dissipate energy at a constant rate is probably wrong because it leads to the conclusion that braking takes more time than it actually does.

My final advice to Mark was to leave *lots of room*. You can see from the table that stopping from 210 mph takes well over a quarter mile of very hard, precise, threshold braking at $1G$!

Part 12

CyberCar, Every Racer's DWIM Car?

The cybernetic DWIM car is coming. DWIM stands for "Do What I Mean."[2] It is a commonplace term in the field of Human-machine Interfaces, and refers to systems that automatically interpret the user's intent from his or her inputs.

Cybernetics (or at least one aspect of it) is the science of unifying humans and machines. The objective of cybernetics is usually to amplify human capability with 'intelligent' machines, but sometimes the objective is the reverse. Most of the work in cybernetics has been under the aegis of defense, for building advanced tanks and aircraft. There is a modest amount of cybernetics in the automotive industry, as well. Anti-lock Braking (ABS), Acceleration Slip Reduction (ASR), Electronic Engine Management, and Automatic Traction Control (ATC) are cybernetic DWIM systems—of a kind—already in production. They all make 'corrections' on the driver's input based on an assumed intention. Steer-by-wire, Continuously Variable Transmissions (CVT), and active suspensions are on the immediate horizon. All these features are part of a distinct trend to automate the driving experience. This month, we take a break from hard physics to look at the better and the worse of increased automation, and we look at one concept of the ultimate result, CyberCar.

Among the research directions in cybernetics are advanced sensors for human inputs. One of the more incredible is a system that reads brain waves and figures out what a fighter pilot wants to do directly from patterns in the waves.

A major challenge in the fighter cockpit is information overload. Pilots have far too many instruments, displays, horns, buzzers, radio channels, and idiot lights competing for their attention. In stressful situations, such as high speed dogfights, the pilot's brain simply ignores inputs beyond its capacity, so the pilot may not hear a critical buzzer or see a critical warning light. In the 'intelligent cockpit,' however, the pilot *consciously* suppresses certain displays and auditory channels, thus reducing sensory clutter. By the same token, the intelligent cockpit must be able to override the pilot's choices and

[2]and the word play on 'dream' was too much to resist.

to put up critical displays and to sound alarms in emergencies. In the reduced clutter of the cockpit, then, it is much less likely that a pilot will miss critical information.

How does the pilot select the displays that he[3] wants to see? The pilot cannot afford the time to scroll through menus like those on a personal computer screen or hunt-and-peck on a button panel like that on an automatic bank teller machine.

There are already sensors that can read a pilot's brain waves and anticipate what he wants to look at next. Before the pilot even consciously knows that he wants to look at a weapon status display, for example, the cybernetic system can infer the intention from his brain waves and pop up the display. If he thinks it is time to look at the radar, before he could speak the command, the system reads his brain waves, pops up the radar display, and puts away the weapon status display.

How does it work? During a training phase, the system reads brain waves and gets explicit commands through a button panel. The system analyzes the brain waves, looking for certain unique features that it can associate with the intention (inferred from the command from the button panel) to see the radar display, and other unique features to associate with the intention to look at weapon status, and so on. The system must be trained individually for each pilot. Later, during operation, whenever the system sees the unique brain wave patterns, it 'knows' what the pilot wants to do.

The implications of technology like this for automobiles is amazing. Already, things like ABS are a kind of rudimentary cybernetics. When a driver stands all over the brake pedal, it is assumed that his intention is to stop, not to skid. The ABS system 'knows,' in a manner of speaking, the driver's intention and manages the physical system of the car to accomplish that goal. So, instead of being a mere mechanical linkage between your foot and the brakes, the brake pedal becomes a kind of intentional, DWIM control. Same goes for traction control and ASR. When the driver is on the gas, the system 'knows' that he wants to go forward, not to spin out or do doughnuts. In the case of TC, the system regulates the torque split to the drive wheels, whether there be two or four. In the case of ASR, the system backs off the throttle when there is wheel spin. Cybernetics again.

ABS, TC, and ASR exist now. What about the future? Consider steer-by-wire. CyberCar, the total cybernetic car, infers the driver's intended

[3]Everywhere, 'he' means 'he or she,' 'his' means 'his or her,' etc.

direction from the steering wheel position. It makes corrections to the actual direction of the steered wheels and to the throttle and brakes much more quickly and smoothly than any driver can do. Coupled with slip angle[4] sensors[1] and inertial guidance systems, perhaps based on miniaturized laser/fiber optic gyros (no moving parts), cybernetic steering, throttle, and brake controls will make up a formidable racing car that could drive a course in practically optimal fashion given only the driver's *desired* racing line.

In an understeering situation, when a car is not turning as much as desired, a common driver mistake is to turn the steering wheel more. That is a mistake, however, only because the driver is treating the steering wheel as an *intentional* control rather than the physical control it actually is. In CyberCar, however, the steering wheel *is* an intentional control. When the driver adds more lock in a corner, CyberCar 'knows' that the driver just wants more steering. Near the limits of adhesion, CyberCar knows that the appropriate *physical* reaction is, in fact, some weight transfer to the front, either by trailing throttle or a little braking, and a little less steering wheel lock. When the fronts hook up again, CyberCar can immediately get back into the throttle and add a little more steering lock, all the while tracking the driver's desires through the intentional steering wheel in the cockpit. Similarly, in an oversteer situation, when the driver gives opposite steering lock, CyberCar knows what to do. First, CyberCar determines whether the condition is trailing throttle oversteer (TTO) or power oversteer (PO). It can do this by monitoring tire loads through suspension deflection and engine torque output over time. In TTO, CyberCar adds a little throttle and countersteers. When the drive wheels hook up again, it modulates the throttle and dials in a little forward lock. In PO, CyberCar gently trails off the throttle and countersteers. All the while, CyberCar monitors driver's intentional inputs and the physical status of the car at the rate of several kilohertz (thousands of times per second).

The very terms 'understeer' and 'oversteer' carry cybernetic implication, for these are terms of intent. Understeer means the car is not steering as much as wanted, and oversteer means it is steering too much.

The above description is within current technology. What if we get *really* fantastic? How about doing away with the steering wheel altogether? CyberCar, version II, knows where the driver wants to go by watching his eyes, and it knows whether to accelerate or brake by watching brain waves. With

[4] Also known as grip angle; see Part 10 of this series.

Virtual Reality and teleoperation, the driver does not even have to be inside the car. The driver, wearing binocular video displays that control incar cameras (or even synthetic computer graphics) *via* head position, sits in a virtual cockpit in the pits.

Now we must ask how much cybernetics is desirable? Autocrossing is, largely, a pure driver skill contest. Wheel-to-wheel racing adds racecraft—drafting, passing, deception, *etc.*—to car control skills. Does it not seem that cybernetics eliminates driver skill as a factor by automating it? Is it not just another way for the 'haves' to beat the 'have-nots' by out-spending them? Drivers who do not have ABS have already complained that it gives their competition an unfair advantage. On the other hand, drivers who *do* have it have complained that it reduces their feel of control and their options while braking. I think they doth protest too much.

In the highest forms of racing, where money is literally no object, cybernetics is already playing a critical role. The clutchless seven speed transmissions of the Williams/Renault team dominated the latter half of the 1991 Formula 1 season. But for some unattributable bad luck, they would have won the driver's championship and the constructor's cup. Carrol Smith, noted racing engineer, has been predicting for years that ABS will show up in Formula 1 as soon as systems can be made small and light enough[2]. It seems inevitable to me that cybernetic systems will give the unfair advantage to those teams most awash in money. However, autocrossers, club racers, and other grass roots competitors will be spared the expense, and the experience of being relieved of the enjoyment of car control, for at least another decade or two.

Acknowledgements

Thanks to Phil Ethier for giving me a few tips on car control that I might be able to teach to CyberCar and to Ginger Clark for bringing slip angle sensors to my attention.

References

[1] Patrick Borthelow. "Sensing Tire Slip Angles At the Racetrack." *Sensors*, September 1991.

[2] Carrol Smith. *Engineer to Win, Prepare to Win, Build to Win.* Classic Motorbooks, P.O. Box 1/RT021, Osceola, WI 54020.

Colophon

Brian Beckman began publishing the Physics of Racing series in June 1990 as a series of articles in his sports car club's newsletter. The articles were originally formatted in LaTeX[5] and typeset using TeXtures on the Apple Macintosh computer.

The articles have been widely distributed as Adobe PostScript files for many years, and were converted to HTML by Robert Keller *circa* 1995 for display on the World Wide Web. They can be seen on Keller's home page at http://members.home.net/rck/phor/.

In August 1999, I undertook a conversion of the Physics of Racing series to Adobe Acrobat PDF format. The original LaTeX sources were converted to the newer LaTeX 2_ε format, and a PDF file was generated using the pdftex package developed by Han The Thanh at Masaryk University, Czech Republic. Graphics were sourced from the GIF files in Keller's Web pages, and converted to PDF and PNG formats using the ImageMagick converter by John Cristy and gif2png by The PNG Development Group.

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[5] LaTeX is a collection of macros for the TeX typesetting system developed by Professor Donald Knuth of Stanford University. TeX is a trademark of the American Mathematical Society.

The Physics of Racing, Part 13: Transients

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Obviously, handling is extremely important in any racing car. In an autocross car, it is critical. A poorly handling car with lots of power will not do well at all on the typical autocross course. A Miata or CRX can usually beat a 60's muscle car like a Pontiac GTO even though the Goat may have four or five times the power. Those cars, while magnificently powerful, were designed for straight-line acceleration at the expense of cornering.

This month, we examine one aspect of handling, that of handling *transient* or short-lived forces. Usually, in motor sports contexts, the word "transient" means short-lived cornering forces as opposed to braking and accelerating forces. In broader contexts, it means any short-lived force.

Transients figure prominently in autocross. Perhaps the epitome of a transient-producing autocross feature is a slalom, which requires a car and driver to flick quickly from left to right and back again. Many courses also feature esses, lane changes, chicanes (dual lane changes), alternating gates, and other variations on the theme. All of these require quick cornering response to transients. Some sports cars, like Elans, MR2's, and X1/9's, are designed specifically to have such quick response. The general rule is that these kinds of cars get you into a corner more quickly than do other kinds. They achieve their response with low weight and low *polar moment of inertia* (PMI). A chief goal of this article is to explain PMI.

Most engineering designs are trade-offs, and designing for quick transient response is no exception. Light weight means, generally, a small engine. Low PMI means, generally, placing the engine as close to the center of mass (CM) as possible. So, many quick-response cars are mid-engined, further constraining engine size. With engine size, we get into another trade-off area: cost versus power. Smaller engines are, generally, less powerful. The cheapest way to get engine power is with size. A big, sloppy, over-the-counter American V8 can cheaply give you 300-400 ft-lb of torque. Getting the same torque from a 1.6 liter four-banger can be very expensive and will put you firmly in the Prepared or Modified ranks. But, a bigger engine is a heavier engine, and that means a beefier (heavier) frame and suspension to support it. Therefore, the cheap way to high torque requires sacrificing some transient response for power. This design approach is typified by Corvettes and Camaros. The general rule is that these kinds of cars get you out of a corner more quickly because of the engine torque.

So, we can divide the sports car universe into the lightweight, quick-response-style camp and the ground-thumping, stump-pulling-style camp. Some cars straddle the boundary and try to be lightweight, with low PMI, and powerful. These cars are usually very expensive because the fundamental design compromises are pushed with exotic materials and great amounts of engineer time. Ordinary cars are usually mostly one or the other. No one can say which design style is "better." Both kinds of car are great fun to drive. There will be some courses on which quick-response type cars will have top times and others on which the V8's will be unbeatable. Fortunately, these two styles of cars are usually in different classes.

Let's back up that discussion with some physics. What is transient response and how does it relate to polar moment of inertia?

Any object resists a change in its state of motion. If it is not moving, it resists moving. If it is moving, it resists stopping or changing direction. The resistance is generically called *inertia*. With straight line motion, inertia has only one aspect: *mass*. Handling is mostly about cornering, however, not about straight-line motion.

Cornering is a change in the direction of motion of a car. In order to change the direction of motion, we must change the direction in which the car is pointing. To do that, we must rotate or *yaw* the car. However, the car will resist yawing because the various parts of the car will resist changing their states of motion. Let's say we are cornering to the right, hence yawing clockwise. The suspension parts and frame and cables and engine *etc. etc.* in the front part of the car will resist veering to the right off their prior straight-line course and the suspension parts and frame and differential and gas tank *etc. etc.* in the rear will resist veering to the left off their prior straight-line course. From this observation, we can 'package' the inertial resistance to yawing of any car into a convenient quantity, the PMI. What follows is a simplified, two-dimensional analysis. The full, three-dimensional case is conceptuallly similar though more complicated mathematically.

It turns out that the general motion of any large object can be broken up into the motion of the center of mass, treated as a small particle, and the rotation of the object about its center of mass. This means that to do dynamical calculations that account for cornering, we must apply Newton's Second Law, $F = ma$, *twice*. First, we apply the law to all masses in the car taken as an aggregate with their positions measured with respect to a fixed point on the ground. Second, we apply the law individually to the massive parts of the car with their positions measured from the CM in the car while it moves.

Let's make a list of all the N parts in the car. Let the variable i run over all the items in the list; let the masses of the parts be m_i, their positions on the X axis of the ground coordinate grid be x_i and their positions on the Y axis be y_i. We summarize the position information with *vector* notation, writing a bold character, $\boldsymbol{r}_i$, for the position of the i-th part. Vector notation saves us from having to write two (or three) sets of equations, one for each coordinate direction on the grid. For many purposes, a vector can be treated like a number in symbolic arithmetic. We must break a vector equation apart into its constituent *component* equations when it's time to do number-crunching.

The (vector) position $\boldsymbol{R}$ of the CM with respect to the ground is just the mass-weighted average over all the parts of the car:

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{M = \sum_{i=1}^{N} m_i} \tag{1}$$

The external forces on the car are also vectors: they have X components and Y components. So, we write the sum of all the forces on the car with a bold $\boldsymbol{F}$. Similarly, the velocity of the CM is a vector. It is the change in $\boldsymbol{R}$ over a small time, dt, divided by the time. This is written

$$\boldsymbol{V} = \frac{d\boldsymbol{R}}{dt} \tag{2}$$

The d/dt notation is called a *derivative*. In turn, the acceleration is a small change in the velocity divided by the time:

$$\boldsymbol{A} = \frac{d\boldsymbol{V}}{dt} = \frac{d^2\boldsymbol{R}}{dt^2} \tag{3}$$

The d^2/dt^2 notation is called a *second derivative* and results from two derivatives in succession.

Newton's Second Law for the CM of the car is then

$$\boldsymbol{F} = M\frac{d^2\boldsymbol{R}}{dt^2} \tag{4}$$

where M is the total mass of all the parts in the car. Simple, eh? This is a *differential equation*, and theoretical physics is overwhelmingly concerned with the solutions of such things. In this case, a solution is finding $\boldsymbol{R}$ given M and $\boldsymbol{F}$. We can also simplify the writing of the equations in general by replacing time-derivative notations with dots: one dot for one time derivative and two dots for two derivatives. We get

$$\boldsymbol{F} = M\ddot{\boldsymbol{R}} \tag{5}$$

Now, we consider the parts of the car separately as they yaw (and pitch and roll) about the CM while remaining firmly attached to the car . Let's write all position variables measured with respect to the coordinate grid fixed in the car with overbars, so the vector position of the i-th mass in our list is $\overline{\boldsymbol{r}}_i$.

However, we don't need to use vectors (in two dimensions), because in pure yawing motion about the CM of the car, the radial distance of each car part from the CM remains fixed and each part has the same yaw angle as the whole car.

Let the yaw angle of the car and its coördinate grid measured against the ground-based, inertial coordinates be θ. As each car part is affected by forces, it moves in a yaw-arc around the CM. A small amount of yaw is written $d\theta$. Each part moves perpendicularly to a line drawn from the part to the CM of the car, and the distance it moves is equal to its radial distance from the CM, $\overline{r}_i$ (nonbold: a number, not a vector), times the little amount of yaw $d\theta$. Divide by the little time over which the motions are measured, and you have the velocity of each car part:

$$\overline{v}_i = \overline{r}_i \frac{d\theta}{dt} = \overline{r}_i \dot{\theta} \tag{6}$$

Now, it's easy to apply Newton's second law. Equate the force on the i-th part, $\overline{F}_i$, to the mass of the part times the acceleration of the part:

$$\overline{F}_i = m_i \overline{r}_i \ddot{\theta} \tag{7}$$

We're almost done with the math, so hang in there. If we multiply equation 7 by $\overline{r}_i$ on both sides, the left-hand side becomes the torque of the forces on the i-th part about the CM:

$$\overline{\Lambda}_i = \overline{r}_i \overline{F}_i = m_i \overline{r}_i^2 \ddot{\theta} \tag{8}$$

Now, if we sum this equation up over all the parts in our list, we can drop the i subscript:

$$\overline{\Lambda} = \left(\sum_{i=1}^{N} m_i \overline{r}_i^2 \right) \ddot{\theta} \tag{9}$$

remembering that all parts have the same $\ddot{\theta}$. The reason for doing this is that resulting equation *looks like* Newton's Second Law, equation 5. If you replace $\sum m_i \overline{r}_i^2$ with a symbol, $\overline{I}$, the equation is identical in form:

$$\overline{\Lambda} = \overline{I} \ddot{\theta} \tag{10}$$

Physicists like to find formal equivalences amongst equations because they can use the same mathematical techniques to solve all of the them. The equivalences also hints at deeper insights into similarities in the Universe.

Ok, if you haven't already guessed it, $\overline{I} = \sum m_i \overline{r}_i^2$ is the polar moment of inertia. To compute it for a given car, we take all the parts in the car, measure their masses and their distances from the CM, square, multiply and add. In practice, this is very difficult. I doubt if PMIs are measured very often, but when they are, it is probably done experimentally: by subjecting the car to known torques and measuring how quickly yaw angle accumulates.

We can also see that, for a given rotational torque, the acceleration of yaw angle is inversely proportional to $\overline{I}$. Thus, we have backed up, from first principles, our statement that cars with low PMI respond more quickly, by yawing, to transient cornering forces than do cars with large PMI. A car with a low PMI is designed so that the heavy parts—primarily the engine—are as close to the CM as possible. Moving the engine even a couple of inches closer to the CM can dramatically decrease the PMI because it varies as the *square* of the distance of parts from the CM. Since equation 10 is formally equivalent to Newton's Second Law, an analogous insight applies to that Law. A car with low mass responds more quickly to forces with straight-line changes in motion just as a car with low PMI responds more quickly to torques with rotational changes in motion.

Why would one design a car with a high PMI? Only to get a big, powerful engine into it that might have to be placed in the front or the rear, far from the CM. So, take your pick. Choose a car with a low PMI that yaws very quickly and give up on some engine power. Or, choose a car with a colossal engine and give up on some handling quickness.

italic

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Physics of Racing, Part 14: Why Smoothness?

Brian Beckman, Ph.D.

5 April 2000

I'm back after a hiatus of nine years. Time does fly, doesn't it? For those counting articles, the last one published was part 12; there is no Part 13.

After such a long time away, it might be worthwhile to repeat the motivation and goals of this "Physics of Racing" series. I am a physicist (the "PhD" after my name is from my Union card). I'm also an active participant in motorsports. It would be almost impossible for me not to use my professional training to analyze my hobby. So, I've been thinking for some time about the physics of racing cars.

Part of the fun for me is to do *totally original* analyses. As such, they won't have the specifics of a hardcore engineering analysis. You can look that up in books by Fred Puhn, William Milliken, and Carrol Smith, amongst many others. I want to find the bare-bones physics behind the engineering–at the risk of bypassing some detail. In sum, I analyse things completely from scratch because:

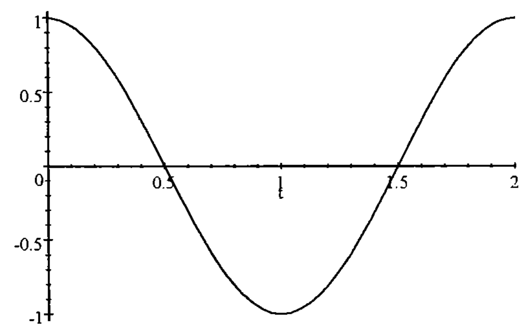
- I want the depth of understanding that can only come only from figuring things out from from first principles,
- "peeking at the answer" from someone else's work would spoil the fun for me,
- I hope to give a somewhat fresh outlook on things.

In 1990, one of my fellow autocrossers asked me to write a monthly column for the SCCA CalClub newsletter. After receiving lots of encouragement, I released the columns to the Internet via Team Dot Net. Back then, the Internet was really small, so I was just sharing the articles in a convenient way with other autocrossers. Since then, the Internet got big and my articles have acquired a life of their own. I have received thousands of happy-customer emails from all over the world, plus a few hate mails (mostly about article #4, in case you're wondering).

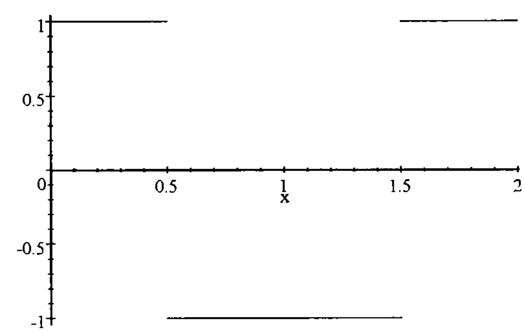
So, here we go again. This month, I'd like to understand, from first principles, why it's so important to be smooth on the controls of a racing car. To me, "smooth" means avoiding jerkiness when applying *or releasing* the brakes, the gas, or steering. Most of the time, you want to roll on and off the gas, squeeze on and off the brakes, slither in and out of steering. It's just as important to avoid jerkiness at the end of a manoever as at the beginning. For example, when steering, not only should you start turning the steering wheel with a gradual, smooth push, but you want to complete the wind-up with a gradual, smooth slowing of the push. Likewise, when you unwind the wheel, you want to start and stop the unwinding smoothly. Thus, a complete steering manoever consists of *four* gradual, slithery start-and-stop mini-manoevers. A complete braking event has four little mini-slithers: one each for the start and stop of the application and the releasing of the pedal. Same for the throttle.

Ok, great, but why? At first blush, it seems one would be able to get back on the gas *more quickly* by snapping the throttle on or get into a corner more quickly by whipping the wheel. Furthermore, supposing we can justify smoothness, are there exceptions to the rule? Are there times when it is best to snap, whip, or jerk? And exactly how smooth should one be? Smoothing means slowing the control inputs down, in a particular way, so it's obviously possible to be too smooth, as in not quick enough, as in not getting as much out of the car as it's capable of delivering.

Let's tackle "why", first. As usual, physics has technical meanings for everyday words. One of the "physically correct" meanings of "smooth" is *sinusoidal.* A sinusoid is a curve that looks like this:



If we think of, say, steering-wheel winding angle as proportional to the vertical axis and time in seconds along the horizontal axis, then this picture describes a really smooth windup taking one second followed immediately by a really smooth unwinding taking another second. In fact, you can easily see the four mini-slithers discussed above as the head-and-tail-sections of the bumps and valleys of the curves. So, the question "why", in technical terms, amounts to asking why such a curve represents better steering input than a curve like the following, "upside-down-hat" curve:



Now, here's the reason: sinusoidal inputs are better because they match the natural response of the car! The suspension and tires perform, approximately, as *damped harmonic oscillators*, or DHOs. A DHO can be in one of three conditions: *underdamped, critically damped*, or *overdamped.* In the underdamped condition, a DHO doesn't have a strong *damper*, which is another term for shock absorber. An underdamped DHO responds sinusoidally. We've all seen cars with broken shocks bouncing up and down on the springs. In the critically damped and overdamped conditions, the car bounces just once, because the damper provides some friction to quiet down continued bouncing. However, even in these conditions, the one bounce has an approximate sinusoidal shape.

The most important parameter of any DHO is its *frequency*. In the underdamped condition, the frequency corresponds to the number of bounces per second the DHO performs. In the critically damped and overdamped conditions–as well as in the underdamped condition, the frequency corresponds to the *resonance* frequency or natural frequency of the system! In other words, if you provide so many inputs per second, back and forth, as in a slalom, at the resonance frequency, the car will have maximal response. If the inputs are faster, they will be too fast for the DHO to catch up and rebound before you've reversed the inputs. If the inputs are slower, the DHO will have caught up and started either to bounce the other way or to settle, depending on condition, when the reverse input comes in.

So here's the bottom line: to maximize the response of a car, you want to provide steering, braking, and throttle inputs with sinusoidal shapes at the resonance frequency of the DHOs that constitute the suspension and tire systems. Inputs that are more jerky just dump high-frequency energy into the system that it must dissipate at lower frequencies. In other words, jerky inputs *upset* the car, which what drivers say all the time. By matching the shape and frequency of your control inputs to the car's natural response curve, you're telling the car to do something it can actually do. By giving the car an "instruction" like the upside-down hat, you're telling it to do something it can't physically do, so it responds by flopping and bouncing around some approximation of your input. Flopping and bouncing means not getting optimum traction; means wasting energy in suspension oscillation; means going slower. Now, there is an exception: if the front tires are *already* sliding, a driver may benefit from quickly steering them into line, hoping to "catch" the car. Likewise, a jerky blip on the throttle with the clutch engaged to bring up the revs to match the gears on a downshift is usually the right thing to do. But, when the car is hooked up, getting the most out of the car means *simulating* the response of the various DHOs in the system with steering, braking, and throttle inputs.

Now we know the physics behind it. Let's do some math!

The frequency turns out to be $\omega = \pm\sqrt{k/m}$, as we show below. k is the *spring constant*, typically measured in pounds per inch, and m is the mass of the sprung weight, typically measured in pound-masses. Suppose our springs were $1,000\,\mathrm{lb}\,/\,\mathrm{in}$, supporting about $800\,\mathrm{lb}$ of weight on one corner of the car. First, we note that a pound *force* is roughly $(1/32)\ slug - \mathrm{ft}\,/\,\mathrm{s}^2$ and that a pound *weight* is $(1/32)\ slug$. So, we're looking at

$$
\begin{aligned}
\omega &= \pm\sqrt{\frac{1,000\frac{\mathrm{lb}-force}{\mathrm{in}}12\frac{\mathrm{in}}{\mathrm{ft}}\frac{1}{32}\frac{slug-\mathrm{ft}\,/\,\mathrm{s}^2}{\mathrm{lb}-force}}{\frac{800\,\mathrm{lb}-weight}{32\,\mathrm{lb}-weight\ /\ slug}}} \\
&= \pm\sqrt{\frac{12,000}{800\,\mathrm{s}^2}} = \pm\sqrt{\frac{120/8}{\mathrm{s}^2}} = \pm\sqrt{\frac{15}{\mathrm{s}^2}} \approx 4/\,\mathrm{s}
\end{aligned}
$$

Notice that we've used the back-of-the-envelope style of computation discussed in part 3 of this series. We've found that the resonance frequency of one corner of a car is about 4 bounces per second! This matches our intuitions and experiences: if one pushes down on the corner of a car with broken shocks, it will bounce up and down a few times a second, not very quickly, not very slowly. We can also see that the frequency varies as the square root of the spring constant. That means that to double the frequency, say, to 8 bounces per second, we must quadruple the spring strength to $4,000\,\mathrm{lb}\,/\,\mathrm{in}$ or quarter the sprung weight to $200\,\mathrm{lb}$. [Note added in proof: My friend, Brad Haase, has pointed out that 4 Hz, while in the "ballpark", is much too fast for a real car. Now, this series of articles is only about fundamental theory and ballpark estimates. Nonetheless, he wrote convincingly "can you imagine a 4-Hz slalom?" I have to admit that 4 Hz seemed too fast to me when I first wrote this article, but I was unable to account for the discrepancy. Brad pointed out that the suspension linkages supply leverage that reduces the effective spring rate and cited the topic "installation ratio" in Milliken's book *Race Car Vehicle Dynamics*. Since I have not peeked at that book, on purpose, as stated in the opening of this entire series and reiterated in this article, I can only confidently refer you there. Nonetheless, intuition says that 1 Hz is more like it, which would argue for an effective spring rate of $1000/16 = 62\,\mathrm{lb}\,/\,\mathrm{in}$.]

How do we derive the frequency formula? Let's work up a sequence of approximations in stages. By improving the approximations gradually, we can check the more advanced approximations for mistakes: they shouldn't be too far off the simple approximations. In the first approximation, ignore the damper, giving us a mass block of sprung weight resting on a spring. This model should act like a corner of a car with a broken shock.

Let the mass of the block be m. The force of gravitation acts downwards on the block with a magnitude mg, where $g = 32.1\,\mathrm{ft}\,/\,\mathrm{s}^2$ is the acceleration of Earth's gravity. The force of the spring acts upward on the mass with a magnitude $k\,(y_0 - y)$, where k is the spring constant and $(y - y_0)$ is the height of the spring above its resting height y_0 (the force term is positive–that is, upward–when $y - y_0$ is negative–that is, when the mass has compressed the spring and the spring pushes back upwards). We can avoid schlepping y_0 around our math by simply defining our coordinate system so that $y_0 = 0$. This sort of trick is very useful in all kinds of physics, even the most advanced.

It's worth noting that the model so far ignores not only the damper, but the weight of the wheel and tire and the spring itself. The weight of the wheel and tire is called the *unsprung weight.* The weight of the spring itself is partially sprung. We don't add these effects in the current article. Today, we stop with just adding the damper back in, below.

Newton's first law guides us from this point on. The total force on the mass is $-ky - mg$. The mass times the acceleration is $m\,(dv_y/dt) = m\,(d^2y/dt^2)$, where v_y is the up-and-down velocity of the mass and dv_y/dt is the rate of change of that velocity. That velocity is, in turn, the rate of change of the y coordinate of the mass block, that is, $v_y = (dy/dt)$. So, the acceleration is the *second* rate of change of y, and we write it as d^2y/dt^2 because that's the way Newton and Leibniz first wrote it 350 years ago. We have the following *dynamic equation* for the motion of our mass block.

$$F = ma \Longrightarrow m\frac{d^2y}{dt^2} = -ky - mg$$

Let's divide the entire equation by m and rearrange it so all the terms are on the left:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y + g = 0$$

If we're careful about units, in particular about $slugs$ and lbs (see article 1), then we can note that k/m has the dimensions of $1/\sec^2$, which is a frequency squared. Let's define

$$\omega^2 = \frac{k}{m}$$

yielding

$$\frac{d^2y}{dt^2} + \omega^2 y + g = 0$$

We need to solve this equation for $y(t)$ as a function of time t. To follow the rest of this, you'll need to know a little freshman calculus. Take, as *ansatz*,

$$y = A + Be^{C\omega t}$$

then

$$\frac{dy}{dt} = C\omega Be^{C\omega t} = C\omega\,(y - A)$$

and

$$\frac{d^2y}{dt^2} = C\omega\frac{dy}{dt} = (C\omega)^2\,(y - A)$$

therefore

$$\begin{aligned}\frac{d^2y}{dt^2} + \omega^2 y + g &= (C\omega)^2\,y - A\,(C\omega)^2 + \omega^2 y + g \\ &= \omega^2\left(C^2 + 1\right)y - \left(A\,(C\omega)^2 - g\right) \\ &= 0 \text{ iff } C^2 = -1 \text{ and } A = -g/\omega^2\end{aligned}$$

So, we see there are two solutions, $y(t) = A + B_1 e^{i\omega t}$ and $y(t) = A + B_2 e^{-i\omega t}$. In fact, the time-dependent parts of these solutions can operate simultaneously, so we *must* write $y(t) = A + B_1 e^{i\omega t} + B_2 e^{-i\omega t}$ in all generality. The values of the two unknowns B_1 and B_2 are determined by two initial conditions, that is, the value of $y(0) = A + B_1 + B_2$ and $(dy/dt)(0) = i\omega\,(B_1 - B_2)$.

Let's get out of the complex domain by writing

$$\begin{aligned}B_1 e^{i\omega t} + B_2 e^{-i\omega t} &= B_1\,(\cos\omega t + i\sin\omega t) + B_2\,(\cos\omega t - i\sin\omega t) \\ &= (B_1 + B_2)\cos\omega t + i\,(B_1 - B_2)\sin\omega t \\ &\triangleq C_1\cos\omega t + C_2\sin\omega t\end{aligned}$$

This definition makes our initial conditions simpler, too:

$$y(0) = C_1;\ v_y(0) = \omega C_2$$

It's easy, now, to add the damper. Damping forces are proportional to the velocity; that is, there is no damping force when things aren't moving. Each corner approximately obeys the equation

$$\frac{d^2y}{dt^2} = -\frac{\delta}{m}\frac{dy}{dt} - \frac{k}{m}y - g$$

where δ is the damper response in lb $-force/\left(\text{ft}\,/\,\text{s}\right)$. The three rightmost terms represent forces, and they are all negative when y and dy/dt are positive. That is, if you pull the sprung weight up, the spring tends to pull it down. Likewise, if the sprung weight is moving up, the damper tends to pull it down. The force of gravitation always pulls the weight down. Let's rewrite, as before:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega^2 y + g = 0$$

where $\omega^2 = k/m$ and $\gamma = \delta/m$. If, as before,

$$y = A + Be^{C\omega t}$$

then

$$\frac{dy}{dt} = C\omega Be^{C\omega t} = C\omega\left(y - A\right)$$

and

$$\frac{d^2y}{dt^2} = C\omega\frac{dy}{dt} = \left(C\omega\right)^2\left(y - A\right)$$

therefore

$$\begin{aligned}\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega^2 y + g &= \left(C\omega\right)^2\left(y - A\right) + C\gamma\omega\left(y - A\right) + \omega^2 y + g \\ &= \left(\left(C^2 + 1\right)\omega + C\gamma\right)\omega y - \left(AC\omega\left(C\omega + \gamma\right) - g\right) \\ &= 0 \text{ iff } \omega C^2 + \gamma C + \omega = 0 \text{ and } AC\omega\left(C\omega + \gamma\right) = g\end{aligned}$$

You may remember the little high-school formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$ for the solution of a quadratic equation. This gives us the answer for C:

$$C = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2\omega}$$

and I'll leave the simple arithmetic for A and the initial conditions to the reader. The critcally damped condition obtains when $\gamma = 2\omega$, overdamped when when $\gamma > 2\omega$, and underdamped when $\gamma < 2\omega$. In the underdamped condition, C has an imaginary component and the exponentials oscillate. Otherwise, they just take one bounce and then settle down.

It will be fun and easy for anyone who has followed along this far to plot out some curves and check out my math. If you find a mistake, please do let me know (I just wrote this off the top of my head, as I always do with these articles).

We could improve the approximation by writing down the coupled equations, that is, treating all four corners of the car together, but that would just be a lot more math without changing the basic physics that the car responds more predictably to smooth inputs and less predictably to jerky inputs. Another improvement would be to add in the effect of the unsprung and partially sprung weight.

Physics of Racing, Part 15:

Bumps In The Road

Brian Beckman, PhD, and Jerry Kuch

This month, we investigate how the effects of road bumps vary with speed. Everyone has experienced that bumps are more punchy as speed increases. A bump that you barely notice at 50 mph can sting at 100 mph. But what about at 200 mph? Will it just smack a little harder, or will it knock your teeth out or, worse, cause you to lose control? Could a bump be the limiting factor in cornering speed? In an aerodynamic car, could a bump cause a sudden and catastrophic loss of downforce and adhesion? To analyze such things, we need an understanding of the variation of bump violence with speed.

At the expense of a little storytelling, let's explain how this topic came up. In particular, where is an amateur motorhead going to have to worry about bumps at 200 mph? At autocrosses, speeds are low, by design, to give everyone a safe venue to challenge the limits. If you're going to spin out, an autocross is the place to do it. Low speed also means, though, that bumps, unless very severe, aren't dominant. On a road course, speeds are higher, as are the consequences of losing control. But speeds are not higher *everywhere*, not for extended times, and seldom approach 200 mph. There are two commonplace scenarios with extended time at high speeds: oval courses and open-road racing. High-speed oval racing is a specialized sport not often encountered by amateurs. Since the focus of this series is on grassroots, amateur hijinks, we'll look at open-road racing.

In Part 11 of this series, we took a scenario for braking from 200 mph from the Silver-State Challenge (SSC) in Nevada. My co-author, Jerry Kuch, and I just ran the 2000 Nevada Open Road Challenge (NORC). This is the May version of the SSC, which is held in September. In all other regards, the NORC and the SCC are the same. For most of the 230 cars entered, these are high-speed, time-speed-distance (TSD) rallies. In each of the sixteen TSD classes, the car running as close as possible to the target speed, over or under, wins. There are TSD classes every five mph from 95 to 170 inclusive, with high and low breakout speeds set by safety concerns. There is also an Unlimited, non-TSD class, in which fastest car wins. This May, the winner of Unlimited averaged 207 mph over a ninety-mile distance and another Unlimited car posted a top speed of 227 mph. Jerry and I ran in the 130-mph class with a top speed of 165 mph.

The SCC and NORC run on a ninety-mile stretch of highway 318 from Lund to Hiko in the Nevada outback, roughly along the shortest path from Twin Falls, ID to Las Vegas. The course runs from north to south, and the road is fabulously stark and beautiful in the unique way of remote desert roads. One is humbled by the realization that if stranded, one would surely perish, probably in a few hours' time, from heat exhaustion, exposure, and dehydration. It's great.

Hwy 318 events have been run continuously on since 1988. In 1990 and 1991, Mark Thornton, a fellow autocrosser, built up his 1986 Super Stock corvette into a Nevada car. Mark and I had nearly identical SS 'vettes, and we often swapped cars at autocrosses. These cars happened to be almost the same as the famous yellow 'vette that Roger Johnson, of multiple SCCA National Championships, still runs in SS, if I'm not mistaken. I know that Roger has driven my car, and I can't recall whether he ever drove Mark's, but I did, many times.

Mark, now deceased, was a bit of a bad boy, and Hwy 318 had just the kind of cachet that appealed to him. The legend goes that the events had been organized by the survivors of the old, illegal 'cannonball' runs. Of course, the NORC and SCC are properly sanctioned and completely legal, despite the fact that they use temporarily closed public highways rather than dedicated race courses.

Not content to play in the TSD classes, Mark decided to convert the black car into an Unlimited machine. I was with Mark when he handed his car off to Dick Guldstrand for blank-check suspension work, and I was in the loop when it went to John Lingenfelter for a reliable engine capable of 200 mph. I met up with Mark in Las Vegas to help with the final preparation of the car. I took a few, tire-warming hops in the car, and, with nearly 600 HP, I can tell you it was seriously fast. Feel free to check out the car's specs at http://www.angelfire.com/wa/brianbec/foober.htm.

Unfortunately, on race day, the car had an oil fire in the first, six-mile straightaway, due to the headers' being a bit too close to the oil-filter canister. The required, on-board halon system saved the car and Mark and I saved what residual fun we could putting it back together and trailering it home. Later that year, Mark won a Triathlon of Motorsports hosted by a hotrodding magazine in the car, and, if I'm not mistaken, repeated the feat in '92. I have been told the car was featured on the cover of the magazine somewhere in those two years, but I have not checked that myself.

I moved to Washington State and lost touch with Mark, who had a non-motorsports accident and passed away. Mark was not uniformly liked, but even his detractors will grant that he was a truly gifted driver and an engaging, entertaining, complex character. Many, currently active autocrossers will remember him.

By sheer, stupid luck, I stumbled across Mark's Nevada car for sale in Florida in 1999. This is about as far away from Seattle as one can get, but the kismet was too much to ignore. I had driven this car many times in anger, had crewed it, was friends with its creator. It just had to come home to me, didn't it? Furthermore, it just HAD to run again in Nevada, didn't it?

I bought the car and began the complex job of preparing it for NORC. One does not contemplate running 200 mph without giving a car a complete checkup. The energy available for destruction at 200 mph is four times the energy available at 100 mph, and sixteen times that available at 50 mph. Furthermore, the car had had an active, open-track life in the intervening years and it was time to tear it down and check it all out. You do NOT want an engine to seize or a suspension part to break at 100 mph, let alone at 200 mph.

With two months to spare, it became obvious that the car would not be ready in time. Better safe than sorry, I asked the mechanics not to hurry and to make sure the car is done *right*. The standards for mechanical work on high-speed cars must be significantly higher than it is for roadgoing and autocross vehicles, for safety. The standards should be comparable to those in aviation. Hurrying is a recognized no-no in aviation, and I applied the same logic to the car work. As I write, I have an ultimate goal of running it in SCC and NORC in ’01 and ’02.

I had already committed to run the ’00 NORC, so I slapped a roll cage in my ’98 Mallett 435 and went on down. This is another fabulous vehicle, but I hadn’t intended to run it in high-speed events until the last minute. It was quite a hustle to get the required safety gear properly installed in time. In hindsight, I don’t regret the decision. The car really came to life at NORC and I’ve run it in several high-speed events since then.

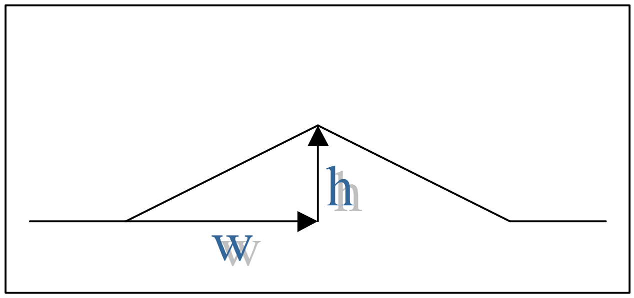
Our flight plan called for holding speeds up to 165 for minutes at a time. As part of planning, we did a survey and calibration run of the course at legal, highway speeds. On the survey run, we noticed several bumpy spots. Driving over them at 70 mph, they were not frightening. But, we had to figure out what to expect at 165. So, right there in the middle of nowhere, we whipped out some envelopes, turned them over, pulled multicolor pens from our pocket protectors, and started scribbling. Geek racing at its best.

Let us take a moment to review the goals and methods of the ‘back-of-the-envelope’ (BOE) style of analysis introduced in Part 3 of this series. Frequently, one simply needs a ballpark estimate or a trend. These are often *much* easier to get than are detailed, precise answers. In fact, they are often easy enough that they can be literally scribbled out on the backs of envelopes *in the field*. And that’s the key point: we needed a rough idea of how the violence of the bumps varies with speed, and we needed it right then and there in the field.

Another benefit of the BOE style is that it can give one a quick plausibility check on numerical data back at the lab. Thoroughgoing engineering analysis usually entails dozens of interlocking equations solved on a computer resulting in tables, plots, and charts. The intuition gets lost in the complexity. It’s sometimes impossible to say, just by looking at a table or chart, whether the results are correct. On the other hand, to get our BOEs, we often make very gross approximations, such as treating the car as a rigid body; or ignoring its track width, that is, treating it as infinitely thin; or ignoring the suspension altogether; or even treating the whole car as a point mass, that is, as if all its mass were concentrated at a single point. Even so, the results are often not wildly off the numerical data, and the discrepancies can usually be explained via non-quantitative arguments. If the BOE and numerical results *are* wildly different, then some detective work is indicated: one or both of them is probably wrong.

BOE is really a semi-quantitative oracle to the physics. These articles are about the physics of racing as opposed to the engineering of racing. We’re primarily interested in the fundamental, theoretical reasons for the behavior of racing cars. The trends and ballpark estimates we get from BOEs often do the job. Of course, this doesn’t mean we won’t get into more detailed treatments and computer simulation. It’s just that we will always be focusing on the physics.

All that said, as usual for BOE, we start with a simplistic model we can solve easily. Think of a bump in the road as a pair of matched triangles, one leading and one trailing.



Let the width of each triangle be w and the height be h. Suppose a car approaches the bump with horizontal speed v. To assess the violence of the bump, let's ask what vertical acceleration the car will experience? If we assume a simplistic model of the car as a rigid body, we get an instantaneous, infinite acceleration right at the instant the car contacts the rising edge. We get further infinite, vertical accelerations at the two other cusps of bump the geometry. However, we know that the tires and suspension will smooth out these sudden impulses. Calculating the effects of tire and suspension flex is too time-consuming to do in the field even if we had data and computers on hand. However, we can get a useful approximation by assuming that the acceleration is distributed over the entire bump.

If the bump is shallow $(h \ll w)$ and the car is fast, then the horizontal speed doesn't change very much and the car goes up the leading edge of the bump in time $t = w/v$. In that time, the car goes upward a distance h, thereby acquiring a vertical speed of $v_y = h/t = vh/w$. Since it acquires that velocity, very roughly, in time t, we can estimate the vertical acceleration to be $a_y \approx v_y/t = h/t^2 = v^2h/w$.

Uh oh. BOE says that the severity of a bump goes up as the *square* of the speed. A bump you can feel at 50 mph is going to be *sixteen times* worse at 200 mph and will most definitely get your attention. The little whoopdeedoos we were noticing at 70 mph would feel $(165/70)^2 = 5.5$ times worse at our planned speed: definitely something to anticipate on-course before we hit them. This BOE also says that the nastiness varies inversely as the width. The wider the bump, the less nasty, linearly. This is plausible.

Now, let's refine the analysis a little. Conservation of energy dictates that the horizontal speed of the car must change. In our simplified, two-dimensional BOE, the velocity vector, $\vec{v}$, consists of two components, horizontal speed, v_x, and vertical speed, v_y. These quantities obey the equation $|\vec{v}|^2 \equiv v^2 = v_x^2 + v_y^2$ whether on the flat or on the bump, that is, no matter what the inclination of the road. We've presupposed, here, that *vertical* always means "in the direction of Earth's gravitation." If we do not change the kinetic energy of the moving car, then $\frac{1}{2}mv^2$ stays constant, therefore v^2 stays constant. On the leading-edge ramp of the bump, remembering trigonometry, $v_x = v\cos(\arctan(h/w)) = vw\big/\sqrt{h^2+w^2}$, $v_y = v\sin(\arctan(h/w)) = vh\big/\sqrt{h^2+w^2}$. Define, as shorthand, $r \equiv \sqrt{h^2+w^2}$, yielding $v_x = vw/r,\ v_y = vh/r$. Using the same approximation as above, we assume that we acquire a vertical velocity of v_y in time $t = w/v_x = wr/vw = r/v$, for a vertical acceleration of

$$a_y \approx \frac{v_y}{t} = \frac{vh/r}{r/v} = \frac{v^2h}{r^2} = \frac{v^2h}{h^2+w^2}$$

This still varies as the square of the speed, we just take a little more time to go over the bump. The only difference to the prior formula, v^2h/w, is the appearance of h^2 in the denominator.

Consider the case of a high, narrow bump. This case was not covered by our first BOE, which assumed that $h = w$. Now, with a high bump, h^2 ? w^2 and $a_y \approx v^2/h$, meaning that the severity of the bump will go *down* linearly with increasing height. Within the confines of our model, this makes sense, because a higher bump gives the car a greater vertical distance in which to suffer its increased vertical velocity, but this doesn't seem *intuitively* correct. A higher bump should be nastier, shouldn't it?

Furthermore, of course, at constant throttle, the kinetic energy of the car *will* change because the force of gravitation will attenuate the vertical velocity. So, in our next consultation of the BOE oracle, we must reduce a_y by $g \approx 32\frac{ft}{s^2}$. The bump is getting less nasty all the time, and it's obvious that we're hitting the limitations of this BOE analysis. To expose the limitations even more starkly, consider two more questions: (1) what about the trailing edge? and (2) what about depressions, that is, down-bumps?

As to the trailing edge, a simplistic car-as-rigid-body would just launch ballistically from the top of the bump. Of course, in a real car, tire elasticity and the suspension would endeavor to keep the tires on the ground. Short of launching, there would just be weight loss causing rebound of the tire sidewalls and the suspension springs. Nevertheless, everyone knows that a ballistic projectile assumes a parabolic flight path, so, as long as the parabola off the top of the bump remains vertically above the down-ramp, our car-as-rigid-body is assured of taking to the air. With the simple bump geometry, we can see that a parabolic launch *always* tarts off above the trailing-edge triangle. It intersects the road again either somewhere on the down-ramp or on the following flat bit of road, depending on horizontal speed.

As to a depression— a down-bump as opposed to an up-bump— a car-as-rigid-body will simply have a ballistic phase before suffering an upward acceleration. At this point, I think we've reached the point of diminishing returns. Let us first repeat that the BOE style is doing what it's supposed to do: getting us rough trends and quantities in the field. Primarily, we wanted to find out how bump severity varies with speed, and we've got our answer: roughly quadratically. We are seeing some ways in which the model departs from intuition and reality and it's time to think about how to improve it back at the lab.

The first point to notice is that we drew a pair of triangles for our bump, but used them only to calculate the time to traverse the bump and the height acquired over that time. This is not a proper *dynamic* analysis, in which we would use Newton's laws to model the motion of the car up and down the bump. At a glance, one can distinguish a dynamic analysis by the presence *mass* in the equations. Nowhere did we use the mass of the car in our BOEs above. Dynamic analysis is often too hard to do in the field because it involves integrating differential equations, almost always by computer.

Another problem concerns our simplistic bump geometry. As noted above, strictly speaking, the severity of a bump on a rigid body *infinite*, no matter what the speed. The reason is that the car acquires its vertical component of velocity instantaneously— in zero time— upon hitting the bump, so the rate of change of the vertical velocity, that is, the vertical acceleration, is infinite at the instant the bump is encountered, then zero on the body of the up-ramp.

Our list-of-things-to-do, should we wish to improve the model, includes the following tasks:

- Model the geometry of the bump more carefully, accounting for the fact that the initiation of the up-ramp, no matter how severe, cannot, in fact, be mathematically instantaneous. Draw some sort of little sinusoidal or exponential curves to account for the actual road profile.

- Integrate the equations motion of the car over the bump.

- Model the car more carefully, accounting for tire flexion, springs, shocks, suspension geometry, mass distribution, moment of inertia, and all the rest. This will entail designing a suspension.

These improvements put us squarely back in the lab. Ultimately, we will resort to computer simulation. As promised years ago, that is the ultimate goal of this series of articles: to spec out a simulation program. Better late than never, right?

Note on Part 14, *Why Smoothness*: The last episode of the *Physics of Racing* sparked a debate on reasonable values for effective wheel spring rates and raised the notion of "installation ratio." The particular point raising the debate was whether 4 Hz was a reasonable value for the resonance frequency of a real racing chassis. It seems it is certainly too fast for a roadgoing car, however, in the time since Part 14 was released I was introduced to a 1980 Group C Ferrari Sports Car. This is essentially a Le Mans car with a lower horsepower engine, for reliability. It is a fully aerodynamic car with ground effects that corners at 2.7g and brakes at 4g. Here's the kicker: its ride height is about half an inch, it does NOT bottom out on bumps, and its spring rate is 14,000 lb/in [sic]. I don't know the installation ratio for this car, but I would be surprised if its chassis resonance frequency was not on the order of 4 Hz or even higher.

Physics of Racing, Part 16:

RARS, A Simple Racing Simulator

Brian Beckman, PhD

If you've been following this series, you know that I've been moving inexorably toward a computer simulation of racing. I've repeatedly debated with myself writing a new one completely from scratch versus starting with someone else's work. Ten years ago, when I started this series, the choice was easy. Since there was nothing out there, I had to start from scratch. The situation has changed. There is at least one competently executed program in the public domain.

Doing a derivative work has undeniable advantages, but conflicts with one of the enduring goals of this series, that is, to do totally original work rather than to recapitulate information you can get from other sources. However, one has to start somewhere. After all, it would be silly for me to rediscover Newton's laws, so I take those as given. Likewise, I've concluded that it would be silly for me to invent the *infrastructure* for a simulation. It would be a very long digression indeed from the Physics of Racing to cover all the groundwork such as

- memory management, windowing, graphics, rendering, data reporting, etc.
- programming languages, scripting, object technology
- simulation technology: time-stepping, eventing, dynamics solvers
- data structures for track description and cars
- arbitrary choices for coordinate systems

All this, while interesting, is not physics. Furthermore, nowadays, it's all more-or-less conventional technology. It's not terribly important for us to make choices in these domains if we can find a competent base platform in which reasonable choices have already been made.

So, with only a little reluctance, I take the decision to start with an existing program. My choice is RARS, the Robot Auto Racing Simulator. This is a lovely, surprisingly simple platform for programmers to experiment with robotics. Its purpose is to support distributed virtual racing competitions, in which entrants write robot drivers and enter them in planned events. The last competitions I have been able to find on the web were conducted in 1999. It is *not* a high-fidelity simulation, and, in fact, was never intended to be. Its physics is simplified in a very clever way to make the main challenge for competitors the writing of robots rather than struggling with elaborate, high-fidelity physics. It supplies a working infrastructure and a large amount of decent data describing famous tracks. Finally, so far as I can tell, RARS is in the public domain.

The simplifications in RARS make it the perfect starting point for enhancing the *physics* without having to reinvent the peripheral aspects of a simulation program. Note that RARS was *designed* for public contribution: the program was originally made to be easy to modify.

The usual mode of modification is for competitors to add new robots. However, it is just as easy to change the physics, as I intend to do. Now, as I take the program in new directions, I will either have to modify the robots or, possibly, create a new, public racing series and throw open the writing of new robots to everyone. Only time will tell what works out best. As usual, however, I will make changes *incrementally*, never deviating very much from the working base. This strategy will not only keep the changes under control, but also enable me to explain to you what's going on, step-by-step.

Therefore, I will create a copy of the sources and change the name of my copy to RARSEP, for "RARS, Enhanced Physics". I will post the source code of my changes on the web to keep the new project rolling along.

My first, long-term goal with RARSEP is to ***find optimal racing lines***. In particular, I need a way to answer questions about racing lines, such as whether the shortest line or the highest-speed line around a particular feature results in the lowest time around the entire course, that is, with the feature in context. Such a question is part of "reading a course", one of the tasks of every racer. In practice, this is a trial-and-error process involving folklore, experience, and experimentation.

For instance, at a recent track day I attended, two instructors, each with many hours on this particular course, were debating a certain combination of slow corners. After quite a bit of haggling and white-board hacking, they agreed that the classic line they *had* been taking for years was probably not the fastest line. It will warm the hearts of autocrossers to find that they had discovered that the autocrosser line, rather than the class road-racer line, was probably fastest.

Autocrossers spend most of their effort finding the fastest way around *slow* corners, whereas the primary challenge for road-course drivers is finding the fastest way around *fast* corners. There is no end of reading material supporting the *classic*, road-racing lines: enter as wide as possible (or, as *high*, as one would say in NASCAR), trail-brake, get back on the throttle in the first half, squeeze on the gas, look up, late apex, and track out. As often as not, however, autocrossers find that simply hugging the inside as tightly— as *low*— as possible yields the quickest way around. Why? Is there science behind this? Can they both be right? How about both wrong? What about intermediate cases: medium-slow and medium-fast corners?

That is an example of the *kind* of question that we want to answer with a simulation program. It was interesting that the instructors' debate concerned slow corners. No one was debating that the fast corners should be taken classically. But, and here I hypothesize, slow corners have the characteristic that ***the corner is not very much larger than the car***. Could it be that when this is true the classic racing line is suboptimal? Experienced autocrossers would, when coming up on such corners, without even thinking about it, go in low and tight and just carry speed or toss-and-catch the car. The instructors had, following the classic theory, been going in high and wide, turning in late, and thereby wasting time trying to form a classic line around corners not much bigger than the car. But, could it be that the classic theory is not best when a corner is so small that the wheelbase of the car is a significant fraction of the distance around the corner? Maybe there are other factors, though. Could it be that the size of the corner does *not* suffice to distinguish an "autocrossy" corner from a “road-racey” corner? Does context matter, as in whether the corner is near other corners or near straights?

It seems that even the most experienced drivers of a particular track will occasionally discover improvements to the line. Some of these improvements depend on transient conditions like weather or the particulars of a certain car or setup. Lots of tracks have canonical “rain lines” that differ from the “dry lines”. I would also bet that Winston Cup cars take different lines around Sears Point and Watkins Glen than do ground-effects sports cars and downforce formula cars. But some improvements will be deep, permanent, invariant revelations that may have eluded the racer on previous outings and analysis. That, in a nutshell, is the first place we're going with RARSEP: to have a way to answer such questions.

I will start with the Windows port of RARS version 074, which you can get in source and binary forms from the following web sites:

- http://users.skynet.be/mgueury/rars/rars.html
- http://www.cgr.ki.se/cgr/persons/mremm/rars/main.htm

I choose the Windows port because it's most convenient for me: I already have working development systems on Windows, whereas to work on other platforms would entail ramp-up time and money. The RARS code base is currently portable to multiple platforms, including Linux and Windows. The code is very well partitioned, so that the platform-dependent bits are separated from the platform-independent bits. Everything I intend to do will be in the platform-independent parts of the program and should build without difficulty on all the platforms. However, I will not be able to *test* my changes on all platforms— the Physics of Racing is not an exercise in industrial-strength, portable software development. While I have no intention of making non-portable changes, there is a small risk that I might inadvertently do so and it could happen some files might someday need a little tweaking to get going on other platforms. I am sure my readers will let me know about it.

The web sites contain very complete descriptions of how to build and run the program, plus how to write robots. To write a robot, one needs to understand the existing physics model of RARS. Similarly, to enhance the physics, we'll need to know the same thing. It presently appears that the best way to enhance the physics incrementally will be in the context of writing a robot, but this may change as we dig in. The subject of this installment of the Physics of Racing will therefore be to introduce the existing RARS physics model along with a long-range plan for enhancing the physics. I am very grateful to the authors for supplying RARS and I hope they will enjoy what I do with their work. The program is very easy to build, run, understand, and enhance. I encourage you to download it and follow along with me. However, my articles will be self-contained: you won't need to build and run RARS to understand what I'm doing with it.

I have found that there is another independent effort afoot to enhance RARS. It's called TORCS and can be found at http://torcs.free.fr. This includes *some* of the enhancements I intend to make, but its goals are like those of RARS rather than like mine. It looks very promising, but it has three features that make it unsuitable as a starting point for me:

- it's unfinished, whereas RARS is functional and established
- it's Linux-based. I don't have a Linux development environment, and it would take me too much time and money to build one up at present
- as usual, peeking (too much) at other work would spoil the fun for me

However, I will be keeping an eye on TORCS. It may turn out to be terrific!

My first approach to adapting RARS to a line-finding task will be to write a robot that learns the optimal line by making small modifications on each lap around the track, much as a human driver would do. This is a kind of *variational* approach, common in physics. The line-finding robot (LFR) will build an internal memory of its current line and everything it discovers about the track. Then, it will tweak the line, and, if the lap time goes down, continue to tweak in the same direction. Otherwise, it will discard the tweak and try another. At the point of diminishing returns, it will start tweaking another part of the line.

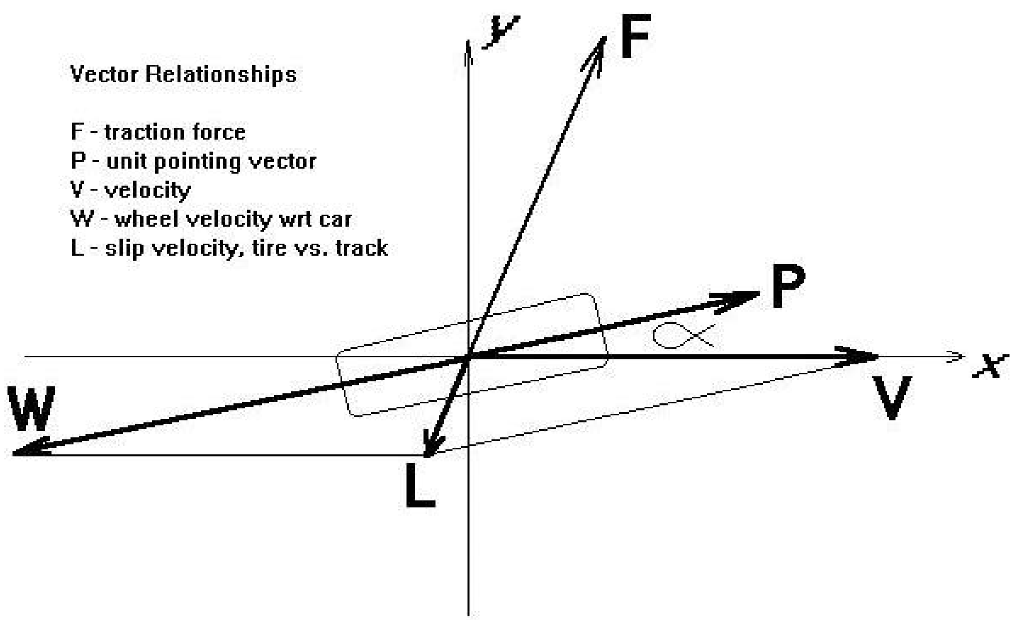
It's going to consume a lot of computing resources and not be competitive in the real-time setting of the old RARS. However, remember, with RARSEP, we are changing the goals. Also, this plan may take considerable time and span many articles. It may not work out at all. As usual, I am taking you along for the ride.

So, let's describe the current physics model. RARS' algorithm is devilishly simple, just the right compromise between physics rich enough to be convincing yet not so complicated that writing a robot is too challenging. Every time step, the simulation engine gives each robot a **situation** structure, and the robot responds with a command or **control** structure. The situation structure contains the current location and velocity of the car relative to the track, the walls, and the other cars. The control structure declares the desired **slip angle**— roughly representative of the steering-wheel angle— and the desired forward velocity— roughly representative of the throttle (positive) and brake (negative). The controls interact with the road through a tire friction model, generating a force that accelerates the car. The force is limited by the power available from the engine, so, it is not always the case that *all* the force the tires *could* deliver can be applied, since the engine may not be able to pump it out. So, the desired velocity may not be the achieved velocity.

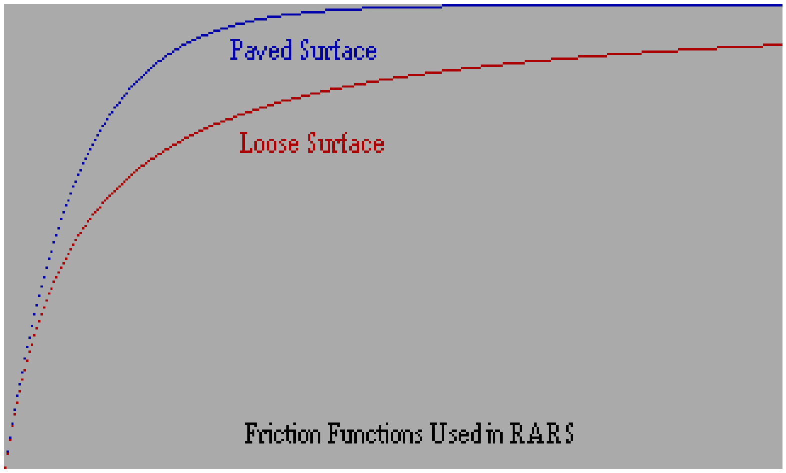
One reason that RARS is simple is that it is two-dimensional. In 2-D RARS, there are three, right-handed coordinate systems. First is the **ground**, a nearly inertial coordinate system fixed with respect to the road. Forces and accelerations are computed in this system, since it is inertial. Second is the **car** coordinate system. The **x-**axis of **car** points forward and the **y-**axis points to driver's left. Third and final is the **path** coordinate system, aligned with the car's velocity vector. The **tangential** component of any vector points along the x-axis of **path**, and the **normal** component of any vector points along the y-axis of **path**. **Car** aligns with **path** only when the car has no slip angle.

The following table, adapted from the program documentation, summarizes the physics model:

V	= car's velocity vector WRT (with respect to) ground
v	= ground speed = magnitude of **V**
P	= forward-pointing unit vector in the **car** system
alpha	= "slip angle" [command output from robot], which separates **P** and **V**. **Alpha** is positive when the car points to the left of **V**, as when power-sliding around a left-hand corner.
W	= velocity vector of tire contact patch WRT **car**, always points backwards along x axis
vc	= "velocity commanded", [command output] forward in the **car** system; **W = -P * vc**
L	= **V** + **W** = **V - P * vc** = "slip vector", velocity of contact patch WRT ground
Lt	= **path**-tangential component of **L** = **v** – **vc** * cos(**alpha**)
Ln	= **path**-normal component of **L** = **- vc** * sin(**alpha**)
l	= slip speed = magnitude of **L**
Q	= **L** / **l** = unit vector in the direction of **L**
mu(**l**)	= coefficient of friction, depending only on slip speed
F	= **- Q** * mass * **mu**(**l**) = force vector pushing the car, in the direction opposite to **L**
f	= mass * **mu**(**l**) = magnitude of **F**
Ft	= **path**-tangential component of **F** = **- f** * **Lt** / **l**
Fn	= **path**-normal component of **F** = **- f** * **Ln** / **l**
FtP	= projection of **Ft** in the **car** system = **Ft** * cos(**alpha**)
FnP	= projection of **Fn** in the **car** system = **Fn** * sin(**alpha**)
pwr	= engine power consumed = sum of force components along **P** limited by engine capacity = max(181hp, (**FtP** + **FnP**) * **vc**)



The friction function currently used is of the form **u(l) = FMAX * l / (K + l)** where **FMAX** and **K** are given constants.



To summarize the limitations of the current model:

- Track
  - Flat, fixed-width, no bumps
- Car
  - Point mass, no suspension

Planned enhancements:

- Track:
  - Elevation changes
  - Width variation
  - Camber, banking
  - Crown, profile
  - FIA berms
  - Bumps
- Car:
  - Four wheels
  - Discrete transmission, gear changes
  - Suspension: springs, dampers
  - Aerodynamics

As we progress, it may be helpful to keep these pages around. We will refer to them frequently.

Physics of Racing, Part 17:

"Slow-in, Fast-out!" or, Advanced Analysis of the Racing Line

Brian Beckman, PhD

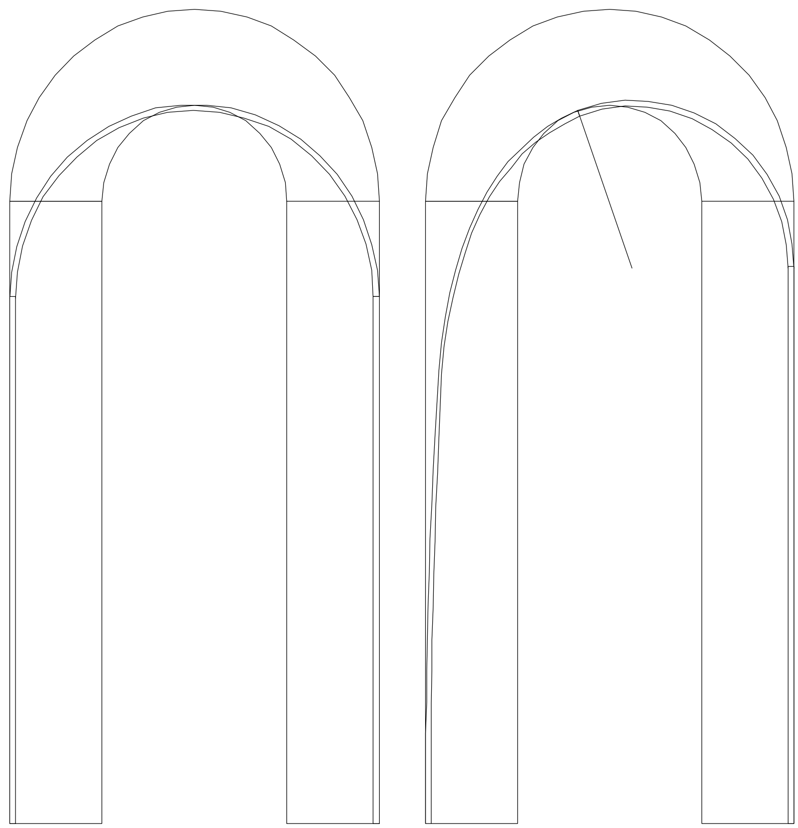
You may remember way back in part 5 that we did some simple calculations by hand to show that the classic racing line through a 90-degree right-hander is better than the either the line that hugs the inside or the line that hugs the outside of the corner. 'Better' means 'has lowest time.' The 'classic racing line' was, under the assumptions of that article, the widest possible inscribed line.

In this and the next installment of *The Physics of Racing*, we raise the bar. Not only do we calculate the times for *all* lines through a corner, but we show a new *kind* of analysis for the exit, accounting for simultaneously accelerating and unwinding the steering wheel after the apex. This kind of analysis requires us to *search* for the lowest time because we cannot calculate it directly. We apply the approximation of the traction circle— subject of part 7— to stay within the capabilities of the car. We also model a more complex segment than in part 5, including an all-important exit chute where we take advantage of improved corner-exit speed. This style of analysis applies directly to computer simulation that we now have in progress in other continuing threads of *The Physics of Racing.*

The whole point of this analysis is to back up the old mantra: "slow-in, fast-out." We will find that the quickest way through the whole segment does *not* include the fastest line around the corner. Rather, **we get the lowest overall time by cornering more slowly so we can get back on the gas earlier.** It's always tempting to corner a little faster, but it frequently does not pay off in the context of the rest of the track.

This analysis is sufficiently long that it will take two installments of this series. In this, the first installment, we do exact calculations on a ***dummy line***, which is the actual line we will drive up to the apex, but just a reference line after the apex. In the next installment, we improve on the dummy line by accelerating and unwinding, predicting the times for a line we would actually drive, but entailing some small inexactitude.

Let's first describe the track segment. Imagine an entry straight of 650 feet, connected to a 180-degree *left-hander* with outer radius 200 feet and inner radius 100 feet, connected to an exit chute of 650 feet. In the following sketch, we show the segment twice with different lines. The line on the left contains the widest possible inscribed cornering radius, and therefore the greatest possible cornering speed. The sketch on the right shows the line with the lowest overall time. Although its cornering speed is slower than in the line on the left, it includes a lengthy acceleration and unwinding phase on exit that more than makes up for it.



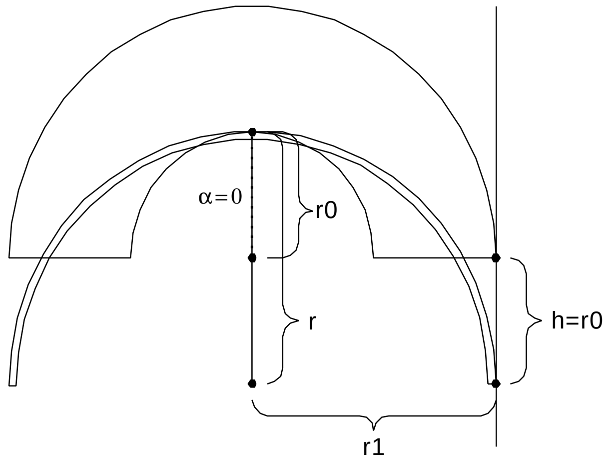
Line with
Fastest Cornering Speed

Line with
Lowest Overall Time

Note that *both* lines begin on the extreme right-hand side of the entry straight. Such will be a feature of every corner we analyze. Lines that begin elsewhere across the entry straight may be valid in scenarios like passing. However, we focus here on lines that are more obvious candidates for lowest times. Also, throughout, we ignore the width of the car, working with the “bicycle line”. If we *were* including the width, w, of the car, we would get the same final results on a track with outer radius of $200 + w/2$ feet and inner radius of $100 - w/2$ feet.

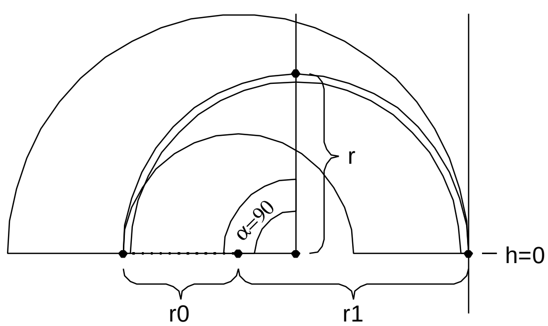
First, we compute exact times where we can on the course: the entry straight, the braking zone, and the corner up to the apex. To have a concrete baseline for comparison, we also do a “suboptimal” exit computation— the dummy line— that includes completing the corner without unwinding and then running down the exit chute dead straight somewhere in the middle of the track. In the next installment of *The Physics of Racing*, we compare the dummy line to the more sophisticated exit that includes simultaneously accelerating and unwinding to use up the entire width of the track in the exit chute.

Let us enter the segment in the right-hand chute at 100 mph = 146.667 fps (feet per second). We want the total times for a number of different cornering radii between two extremes. The largest extreme is a radius of 200 feet, which is the same as the radius of the outer margin of the track. It should be obvious that it is not possible to drive a circle with a radius greater than 200 feet and still stay on the track. This extreme is depicted in the following sketch:



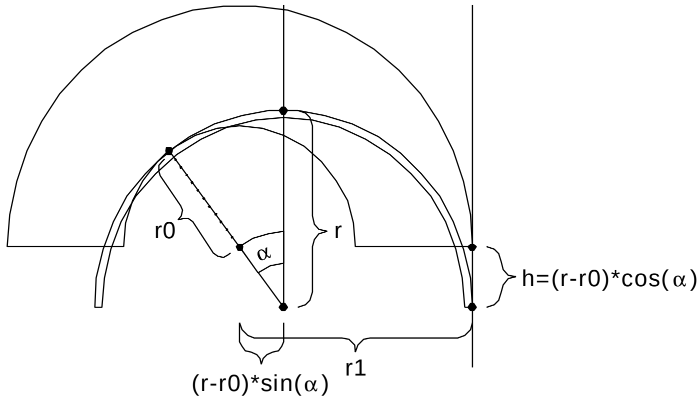
Extreme Case:
Widest Possible Line

We take the opportunity, here, to define a number of parameters that will serve throughout. First, let us call the radius of the outer edge of the track $r1$; this is obviously 200 feet, but, by giving it a symbolic name, we retain the option of changing its numeric value some other time. Likewise, let's call the radius of the inner circle $r0$, now 100 feet. Let's use the symbol r to denote the radius of the inscribed circle we intend to drive. In the extreme case of the widest possible line, r is the same as $r1$, namely, 200 feet. In the other extreme case, that of the tightest inscribed circle, r is 150 feet, as shown in the following sketch:



Extreme Case:
Tightest Possible Line

We're now ready to discuss the two remaining parameters you may have noticed: h and a (Greek letter *alpha*). Consider the following figure illustrating the general case:



General Case, Including the Intermediate Case:
Line with Lowest Overall Time

h indicates the point where we must be done with braking. More precisely, h is the distance of the turn-in point *below* the geometric start of the corner. Its value, by inspection, is $(r-r0)\cos\alpha$. α is the angle past the geometric top where the inscribed circle— the driving line— apexes the inner edge of the track. We see two values for the horizontal distance between the center of the inscribed circle and the center of the inner edge, and those values are $(r-r0)\sin\alpha$ and $r1-r$. Their equality allows us to solve for α:

$$\alpha = \sin^{-1}\left(\frac{r1-r}{r-r0}\right)$$

The following table shows numeric values of h and α for a number of inscribed radii (Note that if we varied $r0$ and $r1$ we would have a much larger 'book' of values to show. For now, we'll just vary r.):

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)
150	90.00	0.00
151	73.90	14.14
152	67.38	20.00
153	62.47	24.49
154	58.41	28.28
155	54.90	31.62
160	41.81	44.72
165	32.58	54.77
170	25.38	63.25
175	19.47	70.71
180	14.48	77.46
185	10.16	83.67
190	6.38	89.44
195	3.02	94.87
200	0.00	100.00

There are a couple of interesting things to notice about these numbers. First, they match up with the visually obvious values of $h=0, \alpha=90$ and $h=100, \alpha=0$ when $r=150, r=200$ respectively. This is a good check that we haven't made a mistake. Secondly, α changes very rapidly with corner radius, and this fact has *major* ramifications on driving line. ***By driving a line just one foot larger than the minimum, one is able to apex more than fifteen degrees later!***

With these data, we're now equipped to compute all the times up to the apex and beyond. First, let's compute the speed in the corner by assuming that our car can corner at $1g = 32.1\ ft/s^2 = v^2/r$, giving us $v=\sqrt{gr}$. We express all speeds in miles per hour, but other lengths in feet. We won't take the time and space to write out all the conversions explicitly, but just remind ourselves once and for all that there are 22 feet per second for every 15 miles per hour.

Now that we have the maximum cornering speed, we can compute how much braking distance we need to get down to that speed from 100 mph. Let's assume that our car can brake at $1g$ also. We know that braking causes us to lose a little velocity for each little increment of time. Precisely, $dv/dt = g$. However, we need to understand how the velocity changes with distance, not with time. Recall that $dx/dt = v, dt = dx/v$, so we get $dx = vdv/g$. Those who remember differential and integral calculus will immediately see that $\Delta x = \frac{1}{2g}\left(v_1^2 - v_2^2\right)$ is the required formula for braking distance. In any event, the braking distance goes as the square of the speed, that is, like the kinetic energy, and that's intuitive. However, there's a factor of two in the numerator that's easy to miss (the origin of this factor is in the calculus, where we compute limit expressions like $\left(v + dv\right)^2 \approx v^2 + 2vdv$).

We next subtract the braking distance from the entry straight, and also subtract h , to give us the distance in which we can go at 100 mph, top speed, before the braking zone.

Now, we need the time spent braking, and that's easy: $\Delta t = \Delta v/g$. All the other times are easy to compute, so here are the times for a variety of cornering lines *up to* the apices (or apexes for those who aren't Latin majors):

Inscribed Corner Radius (ft)	Cornering speed @ 1g in mph	Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Total time (sec) up to the apex
150	47.24	261.11	388.89	2.652	2.418	6.802	11.872
152	47.55	260.11	369.89	2.522	2.404	5.987	10.912
154	47.86	259.11	362.60	2.472	2.390	5.682	10.544
155	48.02	258.61	359.77	2.453	2.382	5.566	10.401
160	48.79	256.11	349.17	2.381	2.347	5.144	9.872
170	50.29	251.11	335.64	2.288	2.278	4.641	9.208
180	51.75	246.11	326.43	2.226	2.212	4.325	8.762
190	53.16	241.11	319.45	2.178	2.147	4.099	8.424
200	54.55	236.11	313.89	2.140	2.083	3.927	8.150

At first glance, it appears that the widest line is a *huge* winner, but we must realize that these times include only driving up to the apex, and that is far earlier on the widest line, where $a = 0$. Suppose we continued driving all the way around the corner at constant speed and then accelerated out the exit chute at 0.5g? This is the dummy line. We won't really drive this line after the apex, but discuss it nonetheless to provide a reference time. It's very easy to compute and provides a foundational intuition for the more advanced exit computation to follow in the next installment:

Inscribed Corner Radius (ft)	Total time (sec) up to the apex	Time (sec) in corner after apex	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time
150	11.872	0.000	11.872	109.091	5.670	17.541	5.670
152	10.912	0.860	11.773	107.857	5.528	17.301	6.388
154	10.544	1.209	11.754	107.422	5.460	17.213	6.669
155	10.401	1.348	11.750	107.260	5.430	17.180	6.779
160	9.872	1.881	11.753	106.697	5.308	17.061	7.189
170	9.208	2.600	11.808	106.101	5.116	16.924	7.716
180	8.762	3.126	11.888	105.806	4.955	16.844	8.082
190	8.424	3.556	11.980	105.666	4.813	16.792	8.369
200	8.150	3.927	12.077	105.627	4.682	16.760	8.609

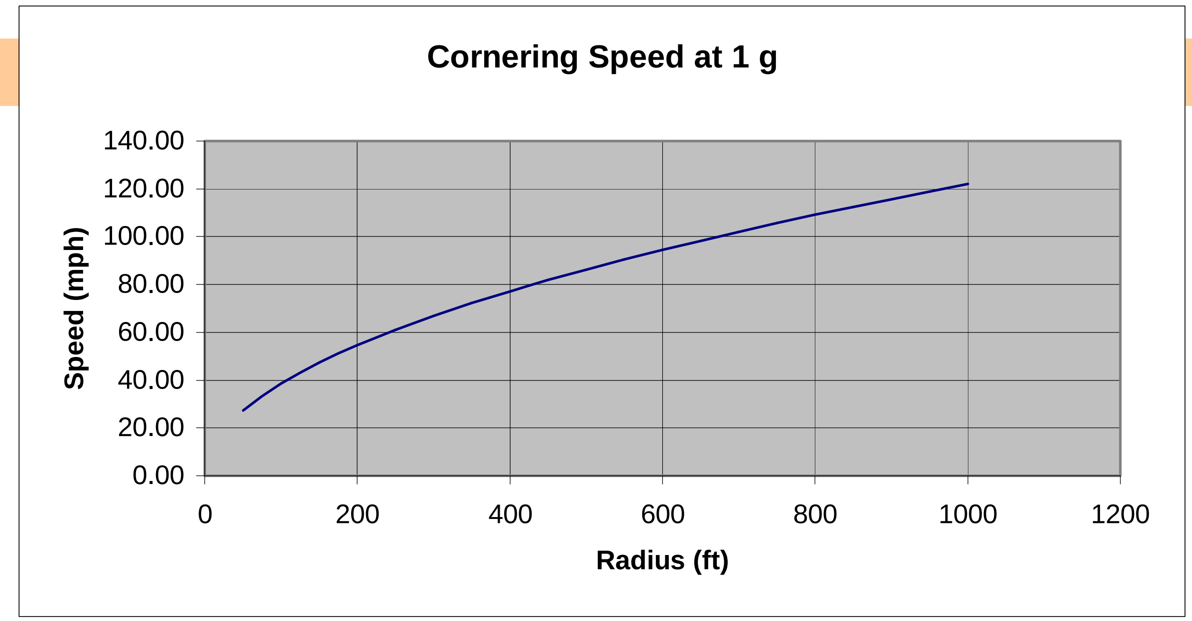
So, we see that, driving the dummy line, the widest line yields the *slowest* time from the entrance up through the complete semicircle, but the quickest *overall* time when the exit chute is included. The widest line has lower (better) times than the tightest line in

- the entry straight by about half a second, because h is large and the entry straight is shorter for wider circles
- in the braking zone by about three tenths because the cornering speed is higher and less braking is needed
- and in the exit chute by almost a second, again because h is large and the exit chute is thereby shorter

The widest line has higher (worse) times by about a second in the circle itself because the wider circle is also longer. When the balances are all added up, the widest line is about eight tenths quicker than the tightest line, but it's ***all because of the effects of the corner on the straights before and after***.

Recall once again that the dummy line is not a line we would actually drive after the apex. But, with that as a framework, we're in position to introduce the next improvement. Everything we do from here on improves just the post-apex portion of the corner and the exit chute. We will actually drive the dummy line up to the apex. So, from this point on, we need only look at the last column in the table above, where we are shocked to see that there are almost three seconds' spread from the slowest to the quickest way out. A good deal of this ought to be available for improvement by accelerating and unwinding.

Radius (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner @ 180 degrees	Time (sec) in exit chute at 1/2 g accel.	Exit speed (mph)	Total time (sec) in segment
50	27.27	311.11	238.89	1.629	3.333	3.927	4.347	74.69	13.236
100	38.57	286.11	263.89	1.799	2.816	5.554	3.753	79.51	13.922
150	47.31	260.06	389.94	2.659	2.407	6.791	5.661	109.26	17.518
200	54.63	235.06	314.94	2.147	2.073	7.842	4.675	105.79	16.737
250	60.98	211.11	338.89	2.311	1.788	8.781	2.888	92.49	15.768
300	66.80	186.11	363.89	2.481	1.521	9.619	2.715	96.42	16.337
350	72.16	161.11	388.89	2.652	1.276	10.390	2.571	100.21	16.889
400	77.14	136.11	413.89	2.822	1.048	11.107	2.449	103.85	17.426
450	81.82	111.11	438.89	2.992	0.833	11.781	2.343	107.37	17.949
500	86.24	86.11	463.89	3.163	0.630	12.418	2.249	110.78	18.461
550	90.45	61.11	488.89	3.333	0.438	13.024	2.167	114.09	18.962
600	94.48	36.11	513.89	3.504	0.253	13.603	2.093	117.30	19.453
650	98.33	11.11	538.89	3.674	0.076	14.159	2.026	120.43	19.935
700	102.05								
750	105.63		fpspmph	1.46667	mphpfps	0.6818182			
800	109.09		fpspmph2	2.15111	mphpfps2	0.4648760			
850	112.45		g	32.1	fpsps				
900	115.71		chute	650	feet				
950	118.88		r_1	200	feet				
1000	121.97		r_0	100	feet				



Start Speed (mph)	End Speed (mph)	Stopping Distance @ 1g (ft)
100	5.00	334.23
100	30.00	304.91
100	55.00	233.71
100	80.00	120.62
100	105.00	-34.34

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Time (sec) in corner after apex	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time	Combined pre-apex time and entry-chute time
150	90.00	0.00	47.31	260.06	389.94	2.659	2.407	6.791	0.000	11.857	109.261	5.661	17.518	5.661	11.857
151	73.90	14.14	47.47	259.56	376.29	2.566	2.400	6.204	0.609	11.780	108.360	5.564	17.344	6.174	11.170
152	67.38	20.00	47.63	259.06	370.94	2.529	2.393	5.977	0.859	11.758	108.025	5.519	17.278	6.378	10.899
153	62.47	24.49	47.78	258.56	366.94	2.502	2.386	5.810	1.049	11.746	107.783	5.483	17.229	6.532	10.698
154	58.41	28.28	47.94	258.06	363.65	2.479	2.379	5.674	1.207	11.739	107.590	5.451	17.190	6.659	10.532
155	54.90	31.62	48.09	257.56	360.81	2.460	2.372	5.557	1.346	11.735	107.428	5.422	17.157	6.768	10.389
160	41.81	44.72	48.86	255.06	350.21	2.388	2.336	5.136	1.878	11.738	106.864	5.300	17.038	7.178	9.860
165	32.58	54.77	49.62	252.56	342.66	2.336	2.302	4.850	2.272	11.761	106.511	5.199	16.959	7.471	9.489
170	25.38	63.25	50.37	250.06	336.69	2.296	2.268	4.634	2.596	11.793	106.267	5.108	16.901	7.704	9.198
175	19.47	70.71	51.10	247.56	331.73	2.262	2.234	4.461	2.874	11.831	106.094	5.025	16.856	7.899	8.957
180	14.48	77.46	51.83	245.06	327.48	2.233	2.201	4.318	3.121	11.873	105.971	4.948	16.821	8.069	8.752
185	10.16	83.67	52.54	242.56	323.77	2.208	2.168	4.197	3.345	11.918	105.886	4.875	16.792	8.220	8.573
190	6.38	89.44	53.25	240.06	320.49	2.185	2.136	4.092	3.551	11.965	105.831	4.805	16.770	8.356	8.414
195	3.02	94.87	53.94	237.56	317.57	2.165	2.104	4.001	3.742	12.013	105.801	4.739	16.751	8.481	8.271
200	0.00	100.00	54.63	235.06	314.94	2.147	2.073	3.921	3.921	12.062	105.792	4.675	16.737	8.596	8.141

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)
150	90.00	0.00
151	73.90	14.14
152	67.38	20.00
153	62.47	24.49
154	58.41	28.28
155	54.90	31.62
160	41.81	44.72
165	32.58	54.77
170	25.38	63.25
175	19.47	70.71
180	14.48	77.46
185	10.16	83.67
190	6.38	89.44
195	3.02	94.87
200	0.00	100.00

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Total time (sec) up to the apex
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	11.184
152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	9.500
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	8.150

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Time (sec) in corner after apex	Total time (sec) in the semi-circle	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time	Combined pre-apex time and entry-chute time
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	0.000	6.802	11.872	109.091	5.670	17.541	5.670	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	0.610	6.824	11.794	108.191	5.573	17.367	6.183	11.184

152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	0.860	6.847	11.773	107.857	5.528	17.301	6.388	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	1.051	6.869	11.761	107.615	5.492	17.252	6.542	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	1.209	6.892	11.754	107.422	5.460	17.213	6.669	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	1.348	6.914	11.750	107.260	5.430	17.180	6.779	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	1.881	7.025	11.753	106.697	5.308	17.061	7.189	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	2.276	7.134	11.776	106.345	5.207	16.982	7.482	9.500
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	2.600	7.241	11.808	106.101	5.116	16.924	7.716	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	2.879	7.347	11.846	105.928	5.033	16.879	7.912	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	3.126	7.451	11.888	105.806	4.955	16.844	8.082	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	3.350	7.554	11.933	105.721	4.882	16.815	8.233	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	3.556	7.655	11.980	105.666	4.813	16.792	8.369	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	3.748	7.755	12.028	105.636	4.746	16.774	8.494	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	3.927	7.854	12.077	105.627	4.682	16.760	8.609	8.150

Braking gs	Cornering gs	Accel gs
1.000	0.00	
0.995	0.10	
0.980	0.20	
0.954	0.30	
0.917	0.40	
0.866	0.50	
0.800	0.60	
0.714	0.70	
0.600	0.80	
0.436	0.90	
0.000	1.00	0.000
	0.90	0.218
	0.80	0.300
	0.70	0.357
	0.60	0.400
	0.50	0.433
	0.40	0.458
	0.30	0.477
	0.20	0.490
	0.10	0.497
	0.00	0.500

t	Inscribed Corner Radius (ft)	alpha	h (ft)	distance to wall (ft)	distance to exit (ft)	heading	speed (mph)	delta wall (feet)	delta exit (feet)	Cornering gs	Accel gs
0	160	41.81	44.72	133.33	724.54	131.81	48.79	1.82	-1.63	1.00	0.000
0.05											

t	r(t)	limiting v(t)
0	150	47.23775
0.05	151	47.39495

cornering gs	accel gs	accel v(t)
1	0.000	47.23775
1	0.000	47.39495

t	a(t) (tangential, fpsps)	v^2/r (radial, fpsps)	a(t) (radial, fpsps)	r(t) (feet)	ax(t) (fpsps)	ay(t) (fpsps)	x(t) (feet)	y(t) (feet)	vx(t) (mph)	vy(t) (mph)	v (mph)
0.00	0.00	32.00	32.00	160.00	-21.33	23.85	66.67	-74.54	36.36	32.52	48.79
0.05	0.32	31.99	31.57	162.27	-21.33	23.27	69.28	-72.09	35.64	33.34	48.80
0.10	0.64	31.97	31.14	164.68	-21.31	22.71	71.84	-69.59	34.91	34.13	48.82
0.15	0.96	31.94	30.70	167.23	-21.26	22.17	74.35	-67.03	34.18	34.91	48.86
0.20	1.28	31.90	30.27	169.92	-21.20	21.64	76.80	-64.41	33.46	35.66	48.90
0.25	1.60	31.84	29.84	172.77	-21.12	21.14	79.20	-61.74	32.74	36.40	48.95
0.30	1.92	31.77	29.41	175.78	-21.01	20.66	81.55	-59.02	32.02	37.12	49.02
0.35	2.24	31.68	28.97	178.95	-20.89	20.20	83.84	-56.25	31.30	37.82	49.09
0.40	2.56	31.59	28.54	182.30	-20.76	19.75	86.09	-53.42	30.59	38.51	49.18
0.45	2.88	31.48	28.11	185.84	-20.61	19.33	88.28	-50.55	29.88	39.19	49.28
0.50	3.20	31.35	27.68	189.56	-20.44	18.93	90.42	-47.63	29.18	39.85	49.39
0.55	3.52	31.22	27.24	193.50	-20.26	18.55	92.51	-44.66	28.48	40.49	49.50
0.60	3.84	31.06	26.81	197.64	-20.06	18.19	94.54	-41.64	27.79	41.12	49.63
0.65	4.16	30.90	26.38	202.01	-19.86	17.85	96.53	-38.58	27.11	41.74	49.77
0.70	4.48	30.72	25.95	206.62	-19.64	17.54	98.47	-35.48	26.43	42.35	49.92
0.75	4.80	30.53	25.51	211.47	-19.41	17.24	100.36	-32.33	25.76	42.95	50.08
0.80	5.12	30.32	25.08	216.59	-19.17	16.96	102.20	-29.13	25.10	43.54	50.25
0.85	5.44	30.09	24.65	221.99	-18.92	16.70	103.99	-25.90	24.44	44.12	50.43
0.90	5.76	29.85	24.22	227.68	-18.67	16.47	105.74	-22.62	23.80	44.69	50.63
0.95	6.08	29.60	23.78	233.68	-18.40	16.25	107.44	-19.30	23.16	45.25	50.83
1.00	6.40	29.33	23.35	240.01	-18.13	16.05	109.09	-15.94	22.53	45.80	51.04
1.05	6.72	29.04	22.92	246.70	-17.85	15.87	110.70	-12.55	21.92	46.35	51.27
1.10	7.04	28.74	22.49	253.75	-17.56	15.71	112.26	-9.11	21.31	46.89	51.50
1.15	7.36	28.41	22.05	261.20	-17.27	15.57	113.78	-5.63	20.71	47.42	51.75
1.20	7.68	28.07	21.62	269.07	-16.97	15.45	115.25	-2.11	20.12	47.96	52.01
1.25	8.00	27.71	21.19	277.39	-16.66	15.34	116.69	1.44	19.54	48.48	52.27
1.30	8.32	27.33	20.76	286.19	-16.35	15.25	118.08	5.04	18.97	49.00	52.55
1.35	8.64	26.93	20.32	295.49	-16.04	15.18	119.43	8.67	18.42	49.52	52.84
1.40	8.96	26.51	19.89	305.34	-15.72	15.13	120.74	12.34	17.87	50.04	53.14
1.45	9.28	26.07	19.46	315.78	-15.40	15.09	122.01	16.05	17.33	50.56	53.45
1.50	9.60	25.60	19.03	326.84	-15.07	15.07	123.24	19.79	16.81	51.07	53.77
1.55	9.92	25.11	18.59	338.57	-14.74	15.06	124.44	23.57	16.30	51.59	54.10
1.60	10.24	24.59	18.16	351.03	-14.41	15.07	125.59	27.39	15.79	52.10	54.44
1.65	10.56	24.04	17.73	364.26	-14.08	15.09	126.72	31.25	15.30	52.61	54.79
1.70	10.88	23.46	17.30	378.34	-13.74	15.13	127.80	35.15	14.82	53.13	55.16
1.75	11.20	22.85	16.86	393.32	-13.40	15.18	128.86	39.08	14.35	53.64	55.53
1.80	11.52	22.21	16.43	409.28	-13.05	15.24	129.88	43.05	13.90	54.16	55.92
1.85	11.84	21.52	16.00	426.31	-12.71	15.32	130.86	47.06	13.45	54.68	56.31
1.90	12.16	20.80	15.57	444.50	-12.36	15.41	131.82	51.11	13.02	55.20	56.72
1.95	12.48	20.02	15.14	463.95	-12.01	15.51	132.74	55.20	12.60	55.73	57.13
2.00	12.80	19.20	14.70	484.77	-11.66	15.62	133.63	59.32	12.19	56.26	57.56
2.05	13.12	18.32	14.27	507.10	-11.31	15.75	134.50	63.49	11.79	56.79	58.00
2.10	13.44	17.36	13.84	531.08	-10.95	15.88	135.33	67.69	11.40	57.33	58.45
2.15	13.76	16.33	13.41	556.88	-10.59	16.03	136.14	71.94	11.03	57.87	58.91
2.20	14.08	15.20	12.97	584.67	-10.23	16.18	136.93	76.22	10.67	58.41	59.38
2.25	14.40	13.95	12.54	614.69	-9.87	16.35	137.68	80.54	10.32	58.97	59.86
2.30	14.72	12.54	12.11	647.16	-9.51	16.52	138.42	84.91	9.99	59.52	60.35
2.35	15.04	10.92	10.92	729.74	-8.39	16.58	139.12	89.32	9.66	60.09	60.86

parameters		
r0	100	feet
r_1	200	feet
r	160	feet
alpha	41.81	degrees
h	44.72	feet
w	100	feet
k	2.5	seconds
v0	48.79	mph
amax	16	fpsps
deltat	0.05	seconds
g	32	fpsps
k_unwind	3.70	seconds

2.40	15.36	8.96	8.96	904.26	-6.51	16.55	139.81	93.76	9.38	60.65	61.37
2.45	15.68	6.37	6.37	1294.17	-3.98	16.45	140.48	98.25	9.15	61.22	61.90
2.50	16.00	0.00	0.00	########	0.00	15.83	141.14	102.78	9.02	61.78	62.43
2.55	16.00	0.00	0.00	#DIV/0!	0.00	15.84	141.81	107.35	9.02	62.32	62.97
2.60	16.00	0.00	0.00	#DIV/0!	0.00	15.84	142.47	111.96	9.02	62.86	63.50
2.65	16.00	0.00	0.00	#DIV/0!	0.00	15.84	143.13	116.61	9.02	63.40	64.03
2.70	16.00	0.00	0.00	#DIV/0!	0.00	15.84	143.79	121.30	9.02	63.94	64.57
2.75	16.00	0.00	0.00	#DIV/0!	0.00	15.85	144.45	126.03	9.02	64.48	65.10
2.80	16.00	0.00	0.00	#DIV/0!	0.00	15.85	145.11	130.80	9.02	65.02	65.64
2.85	16.00	0.00	0.00	#DIV/0!	0.00	15.85	145.77	135.60	9.02	65.56	66.17
2.90	16.00	0.00	0.00	#DIV/0!	0.00	15.85	146.43	140.45	9.02	66.10	66.71
2.95	16.00	0.00	0.00	#DIV/0!	0.00	15.86	147.10	145.34	9.02	66.64	67.24
3.00	16.00	0.00	0.00	#DIV/0!	0.00	15.86	147.76	150.26	9.02	67.18	67.78
3.05	16.00	0.00	0.00	#DIV/0!	0.00	15.86	148.42	155.23	9.02	67.72	68.32
3.10	16.00	0.00	0.00	#DIV/0!	0.00	15.86	149.08	160.24	9.02	68.26	68.85
3.15	16.00	0.00	0.00	#DIV/0!	0.00	15.86	149.74	165.28	9.02	68.80	69.39
3.20	16.00	0.00	0.00	#DIV/0!	0.00	15.87	150.40	170.37	9.02	69.34	69.92
3.25	16.00	0.00	0.00	#DIV/0!	0.00	15.87	151.06	175.49	9.02	69.88	70.46
3.30	16.00	0.00	0.00	#DIV/0!	0.00	15.87	151.72	180.65	9.02	70.42	71.00
3.35	16.00	0.00	0.00	#DIV/0!	0.00	15.87	152.39	185.86	9.02	70.96	71.53
3.40	16.00	0.00	0.00	#DIV/0!	0.00	15.87	153.05	191.10	9.02	71.50	72.07
3.45	16.00	0.00	0.00	#DIV/0!	0.00	15.88	153.71	196.39	9.02	72.05	72.61
3.50	16.00	0.00	0.00	#DIV/0!	0.00	15.88	154.37	201.71	9.02	72.59	73.15
3.55	16.00	0.00	0.00	#DIV/0!	0.00	15.88	155.03	207.07	9.02	73.13	73.68
3.60	16.00	0.00	0.00	#DIV/0!	0.00	15.88	155.69	212.47	9.02	73.67	74.22
3.65	16.00	0.00	0.00	#DIV/0!	0.00	15.88	156.35	217.92	9.02	74.21	74.76
3.70	16.00	0.00	0.00	#DIV/0!	0.00	15.88	157.01	223.40	9.02	74.75	75.29
3.75	16.00	0.00	0.00	#DIV/0!	0.00	15.89	157.68	228.92	9.02	75.29	75.83
3.80	16.00	0.00	0.00	#DIV/0!	0.00	15.89	158.34	234.48	9.02	75.84	76.37
3.85	16.00	0.00	0.00	#DIV/0!	0.00	15.89	159.00	240.08	9.02	76.38	76.91
3.90	16.00	0.00	0.00	#DIV/0!	0.00	15.89	159.66	245.72	9.02	76.92	77.45
3.95	16.00	0.00	0.00	#DIV/0!	0.00	15.89	160.32	251.40	9.02	77.46	77.98
4.00	16.00	0.00	0.00	#DIV/0!	0.00	15.89	160.98	257.12	9.02	78.00	78.52
4.05	16.00	0.00	0.00	#DIV/0!	0.00	15.90	161.64	262.88	9.02	78.54	79.06
4.10	16.00	0.00	0.00	#DIV/0!	0.00	15.90	162.31	268.68	9.02	79.09	79.60
4.15	16.00	0.00	0.00	#DIV/0!	0.00	15.90	162.97	274.52	9.02	79.63	80.14
4.20	16.00	0.00	0.00	#DIV/0!	0.00	15.90	163.63	280.40	9.02	80.17	80.68
4.25	16.00	0.00	0.00	#DIV/0!	0.00	15.90	164.29	286.32	9.02	80.71	81.21
4.30	16.00	0.00	0.00	#DIV/0!	0.00	15.90	164.95	292.28	9.02	81.25	81.75
4.35	16.00	0.00	0.00	#DIV/0!	0.00	15.90	165.61	298.28	9.02	81.80	82.29
4.40	16.00	0.00	0.00	#DIV/0!	0.00	15.90	166.27	304.32	9.02	82.34	82.83
4.45	16.00	0.00	0.00	#DIV/0!	0.00	15.91	166.93	310.39	9.02	82.88	83.37
4.50	16.00	0.00	0.00	#DIV/0!	0.00	15.91	167.60	316.51	9.02	83.42	83.91
4.55	16.00	0.00	0.00	#DIV/0!	0.00	15.91	168.26	322.67	9.02	83.97	84.45
4.60	16.00	0.00	0.00	#DIV/0!	0.00	15.91	168.92	328.87	9.02	84.51	84.99
4.65	16.00	0.00	0.00	#DIV/0!	0.00	15.91	169.58	335.10	9.02	85.05	85.53
4.70	16.00	0.00	0.00	#DIV/0!	0.00	15.91	170.24	341.38	9.02	85.59	86.07
4.75	16.00	0.00	0.00	#DIV/0!	0.00	15.91	170.90	347.70	9.02	86.14	86.61
4.80	16.00	0.00	0.00	#DIV/0!	0.00	15.91	171.56	354.05	9.02	86.68	87.15
4.85	16.00	0.00	0.00	#DIV/0!	0.00	15.92	172.22	360.45	9.02	87.22	87.69
4.90	16.00	0.00	0.00	#DIV/0!	0.00	15.92	172.89	366.88	9.02	87.76	88.22

4.95	16.00	0.00	0.00	#DIV/0!	0.00	15.92	173.55	373.36	9.02	88.31	88.76
5.00	16.00	0.00	0.00	#DIV/0!	0.00	15.92	174.21	379.88	9.02	88.85	89.30
5.05	16.00	0.00	0.00	#DIV/0!	0.00	15.92	174.87	386.43	9.02	89.39	89.84
5.10	16.00	0.00	0.00	#DIV/0!	0.00	15.92	175.53	393.03	9.02	89.93	90.38
5.15	16.00	0.00	0.00	#DIV/0!	0.00	15.92	176.19	399.66	9.02	90.48	90.92
5.20	16.00	0.00	0.00	#DIV/0!	0.00	15.92	176.85	406.34	9.02	91.02	91.46
5.25	16.00	0.00	0.00	#DIV/0!	0.00	15.92	177.51	413.05	9.02	91.56	92.00
5.30	16.00	0.00	0.00	#DIV/0!	0.00	15.92	178.18	419.80	9.02	92.10	92.54
5.35	16.00	0.00	0.00	#DIV/0!	0.00	15.92	178.84	426.60	9.02	92.65	93.09
5.40	16.00	0.00	0.00	#DIV/0!	0.00	15.93	179.50	433.43	9.02	93.19	93.63
5.45	16.00	0.00	0.00	#DIV/0!	0.00	15.93	180.16	440.31	9.02	93.73	94.17
5.50	16.00	0.00	0.00	#DIV/0!	0.00	15.93	180.82	447.22	9.02	94.28	94.71
5.55	16.00	0.00	0.00	#DIV/0!	0.00	15.93	181.48	454.17	9.02	94.82	95.25
5.60	16.00	0.00	0.00	#DIV/0!	0.00	15.93	182.14	461.17	9.02	95.36	95.79
5.65	16.00	0.00	0.00	#DIV/0!	0.00	15.93	182.80	468.20	9.02	95.91	96.33
5.70	16.00	0.00	0.00	#DIV/0!	0.00	15.93	183.47	475.27	9.02	96.45	96.87
5.75	16.00	0.00	0.00	#DIV/0!	0.00	15.93	184.13	482.39	9.02	96.99	97.41
5.80	16.00	0.00	0.00	#DIV/0!	0.00	15.93	184.79	489.54	9.02	97.53	97.95
5.85	16.00	0.00	0.00	#DIV/0!	0.00	15.93	185.45	496.73	9.02	98.08	98.49
5.90	16.00	0.00	0.00	#DIV/0!	0.00	15.93	186.11	503.96	9.02	98.62	99.03
5.95	16.00	0.00	0.00	#DIV/0!	0.00	15.93	186.77	511.23	9.02	99.16	99.57
6.00	16.00	0.00	0.00	#DIV/0!	0.00	15.93	187.43	518.55	9.02	99.71	100.11
6.05	16.00	0.00	0.00	#DIV/0!	0.00	15.94	188.10	525.90	9.02	100.25	100.65
6.10	16.00	0.00	0.00	#DIV/0!	0.00	15.94	188.76	533.29	9.02	100.79	101.20
6.15	16.00	0.00	0.00	#DIV/0!	0.00	15.94	189.42	540.72	9.02	101.34	101.74
6.20	16.00	0.00	0.00	#DIV/0!	0.00	15.94	190.08	548.19	9.02	101.88	102.28
6.25	16.00	0.00	0.00	#DIV/0!	0.00	15.94	190.74	555.70	9.02	102.42	102.82
6.30	16.00	0.00	0.00	#DIV/0!	0.00	15.94	191.40	563.25	9.02	102.97	103.36
6.35	16.00	0.00	0.00	#DIV/0!	0.00	15.94	192.06	570.84	9.02	103.51	103.90
6.40	16.00	0.00	0.00	#DIV/0!	0.00	15.94	192.72	578.48	9.02	104.05	104.44
6.45	16.00	0.00	0.00	#DIV/0!	0.00	15.94	193.39	586.15	9.02	104.60	104.98
6.50	16.00	0.00	0.00	#DIV/0!	0.00	15.94	194.05	593.86	9.02	105.14	105.53
6.55	16.00	0.00	0.00	#DIV/0!	0.00	15.94	194.71	601.61	9.02	105.68	106.07
6.60	16.00	0.00	0.00	#DIV/0!	0.00	15.94	195.37	609.40	9.02	106.23	106.61
6.65	16.00	0.00	0.00	#DIV/0!	0.00	15.94	196.03	617.23	9.02	106.77	107.15
6.70	16.00	0.00	0.00	#DIV/0!	0.00	15.94	196.69	625.10	9.02	107.31	107.69
6.75	16.00	0.00	0.00	#DIV/0!	0.00	15.94	197.35	633.01	9.02	107.86	108.23
6.80	16.00	0.00	0.00	#DIV/0!	0.00	15.94	198.01	640.95	9.02	108.40	108.78
6.85	16.00	0.00	0.00	#DIV/0!	0.00	15.95	198.68	648.94	9.02	108.95	109.32
6.90	16.00	0.00	0.00	#DIV/0!	0.00	15.95	199.34	656.97	9.02	109.49	109.86
6.95	16.00	0.00	0.00	#DIV/0!	0.00	15.95	200.00	665.04	9.02	110.03	110.40
7.00	16.00	0.00	0.00	#DIV/0!	0.00	15.95	200.66	673.15	9.02	110.58	110.94
7.05	16.00	0.00	0.00	#DIV/0!	0.00	15.95	201.32	681.30	9.02	111.12	111.48
7.10	16.00	0.00	0.00	#DIV/0!	0.00	15.95	201.98	689.49	9.02	111.66	112.03
7.15	16.00	0.00	0.00	#DIV/0!	0.00	15.95	202.64	697.72	9.02	112.21	112.57
7.20	16.00	0.00	0.00	#DIV/0!	0.00	15.95	203.30	705.99	9.02	112.75	113.11
7.25	16.00	0.00	0.00	#DIV/0!	0.00	15.95	203.97	714.29	9.02	113.29	113.65
7.30	16.00	0.00	0.00	#DIV/0!	0.00	15.95	204.63	722.64	9.02	113.84	114.19
7.35	16.00	0.00	0.00	#DIV/0!	0.00	15.95	205.29	731.03	9.02	114.38	114.74
7.40	16.00	0.00	0.00	#DIV/0!	0.00	15.95	205.95	739.46	9.02	114.93	115.28
7.45	16.00	0.00	0.00	#DIV/0!	0.00	15.95	206.61	747.93	9.02	115.47	115.82

7.50	16.00	0.00	0.00	#DIV/0!	0.00	15.95	207.27	756.43	9.02	116.01	116.36
7.55	16.00	0.00	0.00	#DIV/0!	0.00	15.95	207.93	764.98	9.02	116.56	116.91
7.60	16.00	0.00	0.00	#DIV/0!	0.00	15.95	208.59	773.57	9.02	117.10	117.45
7.65	16.00	0.00	0.00	#DIV/0!	0.00	15.95	209.26	782.20	9.02	117.64	117.99
7.70	16.00	0.00	0.00	#DIV/0!	0.00	15.95	209.92	790.86	9.02	118.19	118.53
7.75	16.00	0.00	0.00	#DIV/0!	0.00	15.95	210.58	799.57	9.02	118.73	119.07
7.80	16.00	0.00	0.00	#DIV/0!	0.00	15.95	211.24	808.32	9.02	119.28	119.62
7.85	16.00	0.00	0.00	#DIV/0!	0.00	15.95	211.90	817.10	9.02	119.82	120.16
7.90	16.00	0.00	0.00	#DIV/0!	0.00	15.96	212.56	825.93	9.02	120.36	120.70
7.95	16.00	0.00	0.00	#DIV/0!	0.00	15.96	213.22	834.80	9.02	120.91	121.24
8.00	16.00	0.00	0.00	#DIV/0!	0.00	15.96	213.89	843.70	9.02	121.45	121.79
8.05	16.00	0.00	0.00	#DIV/0!	0.00	15.96	214.55	852.65	9.02	122.00	122.33
8.10	16.00	0.00	0.00	#DIV/0!	0.00	15.96	215.21	861.64	9.02	122.54	122.87
8.15	16.00	0.00	0.00	#DIV/0!	0.00	15.96	215.87	870.66	9.02	123.08	123.41
8.20	16.00	0.00	0.00	#DIV/0!	0.00	15.96	216.53	879.73	9.02	123.63	123.96
8.25	16.00	0.00	0.00	#DIV/0!	0.00	15.96	217.19	888.83	9.02	124.17	124.50
8.30	16.00	0.00	0.00	#DIV/0!	0.00	15.96	217.85	897.98	9.02	124.72	125.04
8.35	16.00	0.00	0.00	#DIV/0!	0.00	15.96	218.51	907.17	9.02	125.26	125.58
8.40	16.00	0.00	0.00	#DIV/0!	0.00	15.96	219.18	916.39	9.02	125.80	126.13
8.45	16.00	0.00	0.00	#DIV/0!	0.00	15.96	219.84	925.66	9.02	126.35	126.67
8.50	16.00	0.00	0.00	#DIV/0!	0.00	15.96	220.50	934.96	9.02	126.89	127.21
8.55	16.00	0.00	0.00	#DIV/0!	0.00	15.96	221.16	944.31	9.02	127.44	127.75
8.60	16.00	0.00	0.00	#DIV/0!	0.00	15.96	221.82	953.69	9.02	127.98	128.30
8.65	16.00	0.00	0.00	#DIV/0!	0.00	15.96	222.48	963.12	9.02	128.52	128.84
8.70	16.00	0.00	0.00	#DIV/0!	0.00	15.96	223.14	972.58	9.02	129.07	129.38
8.75	16.00	0.00	0.00	#DIV/0!	0.00	15.96	223.80	982.09	9.02	129.61	129.93
8.80	16.00	0.00	0.00	#DIV/0!	0.00	15.96	224.47	991.63	9.02	130.16	130.47
8.85	16.00	0.00	0.00	#DIV/0!	0.00	15.96	225.13	1001.22	9.02	130.70	131.01
8.90	16.00	0.00	0.00	#DIV/0!	0.00	15.96	225.79	1010.84	9.02	131.25	131.55
8.95	16.00	0.00	0.00	#DIV/0!	0.00	15.96	226.45	1020.51	9.02	131.79	132.10
9.00	16.00	0.00	0.00	#DIV/0!	0.00	15.96	227.11	1030.21	9.02	132.33	132.64
9.05	16.00	0.00	0.00	#DIV/0!	0.00	15.96	227.77	1039.95	9.02	132.88	133.18
9.10	16.00	0.00	0.00	#DIV/0!	0.00	15.96	228.43	1049.74	9.02	133.42	133.73
9.15	16.00	0.00	0.00	#DIV/0!	0.00	15.96	229.09	1059.56	9.02	133.97	134.27
9.20	16.00	0.00	0.00	#DIV/0!	0.00	15.96	229.76	1069.43	9.02	134.51	134.81
9.25	16.00	0.00	0.00	#DIV/0!	0.00	15.96	230.42	1079.33	9.02	135.05	135.36
9.30	16.00	0.00	0.00	#DIV/0!	0.00	15.96	231.08	1089.28	9.02	135.60	135.90
9.35	16.00	0.00	0.00	#DIV/0!	0.00	15.97	231.74	1099.26	9.02	136.14	136.44
9.40	16.00	0.00	0.00	#DIV/0!	0.00	15.97	232.40	1109.28	9.02	136.69	136.98
9.45	16.00	0.00	0.00	#DIV/0!	0.00	15.97	233.06	1119.35	9.02	137.23	137.53
9.50	16.00	0.00	0.00	#DIV/0!	0.00	15.97	233.72	1129.45	9.02	137.78	138.07
9.55	16.00	0.00	0.00	#DIV/0!	0.00	15.97	234.38	1139.59	9.02	138.32	138.61
9.60	16.00	0.00	0.00	#DIV/0!	0.00	15.97	235.05	1149.78	9.02	138.86	139.16
9.65	16.00	0.00	0.00	#DIV/0!	0.00	15.97	235.71	1160.00	9.02	139.41	139.70
9.70	16.00	0.00	0.00	#DIV/0!	0.00	15.97	236.37	1170.26	9.02	139.95	140.24
9.75	16.00	0.00	0.00	#DIV/0!	0.00	15.97	237.03	1180.57	9.02	140.50	140.79
9.80	16.00	0.00	0.00	#DIV/0!	0.00	15.97	237.69	1190.91	9.02	141.04	141.33
9.85	16.00	0.00	0.00	#DIV/0!	0.00	15.97	238.35	1201.29	9.02	141.59	141.87
9.90	16.00	0.00	0.00	#DIV/0!	0.00	15.97	239.01	1211.72	9.02	142.13	142.42
9.95	16.00	0.00	0.00	#DIV/0!	0.00	15.97	239.68	1222.18	9.02	142.67	142.96
10.00	16.00	0.00	0.00	#DIV/0!	0.00	15.97	240.34	1232.68	9.02	143.22	143.50

10.05	16.00	0.00	0.00	#DIV/0!	0.00	15.97	241.00	1243.22	9.02	143.76	144.05
10.10	16.00	0.00	0.00	#DIV/0!	0.00	15.97	241.66	1253.81	9.02	144.31	144.59
10.15	16.00	0.00	0.00	#DIV/0!	0.00	15.97	242.32	1264.43	9.02	144.85	145.13
10.20	16.00	0.00	0.00	#DIV/0!	0.00	15.97	242.98	1275.09	9.02	145.40	145.68
10.25	16.00	0.00	0.00	#DIV/0!	0.00	15.97	243.64	1285.79	9.02	145.94	146.22
10.30	16.00	0.00	0.00	#DIV/0!	0.00	15.97	244.30	1296.54	9.02	146.49	146.76
10.35	16.00	0.00	0.00	#DIV/0!	0.00	15.97	244.97	1307.32	9.02	147.03	147.31
10.40	16.00	0.00	0.00	#DIV/0!	0.00	15.97	245.63	1318.14	9.02	147.57	147.85
10.45	16.00	0.00	0.00	#DIV/0!	0.00	15.97	246.29	1329.00	9.02	148.12	148.39
10.50	16.00	0.00	0.00	#DIV/0!	0.00	15.97	246.95	1339.90	9.02	148.66	148.94
10.55	16.00	0.00	0.00	#DIV/0!	0.00	15.97	247.61	1350.85	9.02	149.21	149.48
10.60	16.00	0.00	0.00	#DIV/0!	0.00	15.97	248.27	1361.83	9.02	149.75	150.02
10.65	16.00	0.00	0.00	#DIV/0!	0.00	15.97	248.93	1372.85	9.02	150.30	150.57
10.70	16.00	0.00	0.00	#DIV/0!	0.00	15.97	249.59	1383.91	9.02	150.84	151.11
10.75	16.00	0.00	0.00	#DIV/0!	0.00	15.97	250.26	1395.01	9.02	151.39	151.65
10.80	16.00	0.00	0.00	#DIV/0!	0.00	15.97	250.92	1406.15	9.02	151.93	152.20
10.85	16.00	0.00	0.00	#DIV/0!	0.00	15.97	251.58	1417.34	9.02	152.47	152.74
10.90	16.00	0.00	0.00	#DIV/0!	0.00	15.97	252.24	1428.56	9.02	153.02	153.28
10.95	16.00	0.00	0.00	#DIV/0!	0.00	15.97	252.90	1439.82	9.02	153.56	153.83

Physics of Racing, Part 18:

"Slow In, Fast Out!" or, Advanced Racing Line, Continued

Brian Beckman, PhD

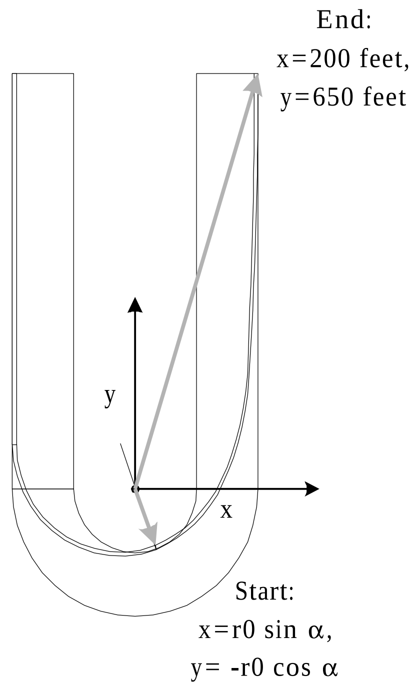
In the previous installment, we did exact calculations for a dummy line down a 650-foot entry straight, a 180-degree left-hander, and a 650-foot exit chute. Cornering radii vary from 150 feet to 200 feet, and the track is 100 feet wide all the way around. This dummy line carries constant speed around the entire left-hander. We did those calculations to provide reference times to compare against this month's more sophisticated calculations, in which we unwind the steering wheel and accelerate at the same time. The baseline times for the dummy line over the whole course, as a function of cornering radius, are in the second-to-last column of the following table:

Inscribed Corner Radius (ft)	Total time (sec) up to the apex	Time (sec) in corner after apex	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time
150	11.872	0.000	11.872	109.091	5.670	17.541	5.670
152	10.912	0.860	11.773	107.857	5.528	17.301	6.388
154	10.544	1.209	11.754	107.422	5.460	17.213	6.669
155	10.401	1.348	11.750	107.260	5.430	17.180	6.779
160	9.872	1.881	11.753	106.697	5.308	17.061	7.189
170	9.208	2.600	11.808	106.101	5.116	16.924	7.716
180	8.762	3.126	11.888	105.806	4.955	16.844	8.082
190	8.424	3.556	11.980	105.666	4.813	16.792	8.369
200	8.150	3.927	12.077	105.627	4.682	16.760	8.609

From this point on, we need only look at the last column. It's after the apex and down the exit chute where we look for improvement; we actually drive the dummy line up to the apex. Many readers will be screaming that we *could* try to get on the gas *before* the apex for even *more* improvement. Others will be screaming "trail brake!," that is, ease off the brakes at the same time as winding the steering wheel at turn in (thanks to reader Marc Sibilia for pointing this out to me). We leave those refinements to later articles.

The approach in this article is to find a line by building it up, step-by-step, honoring the traction circle and the sides of the track. This is one of the techniques we can use in computer simulations, so we get to kill two birds with one stone: previewing simulation and analyzing a particular driving line. For convenience, we need a Cartesian coordinate system, that is, a square grid. Let's turn the track around 180 degrees for this purpose, and put the center of the coordinate system at the center of the corner. Since the inside edge of the track and the outside edge of the track are concentric semicircles, there is only one identifiable center of the corner.

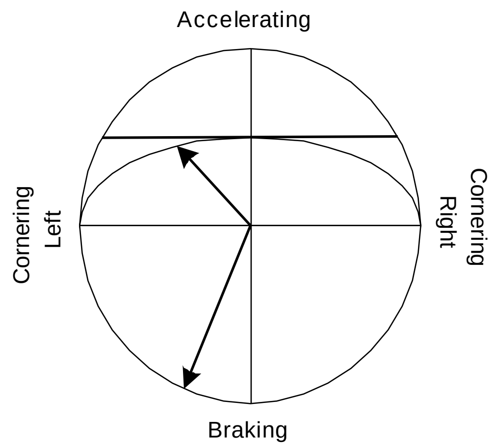
We'll work by measuring the position and heading of the centroid of the car with respect to this new coordinate system. We have a goal of arriving at the point $x = 200,\ y = 650$, measured in feet, in the least possible time, with a heading of as close to 90 degrees as we can get it, that is, heading straight down the track. We start at the apex, which measures from $x = r_0 \sin\alpha,\ y = -r_0 \cos\alpha$. The following sketch illustrates:



I must note, at this point, if you haven't already noticed, this installment of *The Physics of Racing* is going to be more concentrated and intense than previous installments. I'm just going to blurt out facts without the usual explanations and walkthroughs. The reasons are (1) that we have a lot to get through in a little space and (2) that we assume that if you've been following the series this far, you've got the fortitude to work through it. So, *let's get it on*!

The initial heading is tangent to the inner edge of the track, that is, perpendicular to the line from the center of the track's corner to the apex. Therefore, it has the angle α up from the horizontal x axis. We know the starting speed, $v0$, so we know its components in the x direction and in the y direction: $v0_x = v0\cos\alpha,\ v0_y = v0\sin\alpha$.

We perform the entire maneuver whilst never exceeding the limits of the traction circle. We set those limits as 1g cornering and braking and 0.5g accelerating, with smooth transitions all way around, as in the following sketch (the horizontal cap shows a way of accounting for engine limitations with *non*-smooth transitions, which will allow us to accelerate harder with the wheel still turned but probably scare us in the seat. Also, we note that 0.5g is a plausible, if only approximate, number for acceleration. We leave it to the reader to show that 0.5g in the quarter mile results in a realistic 13-second elapsed time, if at an unrealistic speed of 150 mph):



The Traction Circle

In each step of the calculation, we keep track of the following information:

- the time, t
- the current position, $x(t), y(t)$, which we check to make sure we're still on the track $(x < 200)$ and to see whether we're done $(y \geq 650)$
- the current velocity, $v_x(t), v_y(t)$, which we use to update the current position: $x(t + \Delta t) = x(t) + v(t) * \Delta t$, and likewise for y
- the tangential and radial acceleration, $a_t(t), a_r(t)$, that is, tangential and radial to the bit of racing line at each instant (the *instantaneous* line), which we check to make sure that we're not cornering over the limit and that we're not exceeding the capacity of the engine, i.e., that $\sqrt{a_t^2 + a_r^2}$ is inside the traction envelope
- the acceleration in the x and y directions, $a_x(t), a_y(t)$, which we use to update the current velocity: $v_x(t + \Delta t) = v_x(t) + a_x(t) * \Delta t$, and likewise for v_y

We drive the whole simulation by feeding on the throttle linearly with time over a time span called k and by simultaneously increasing the instantaneous radius of the driving line over a potentially different time span called k_{unwind}. Feeding on the throttle allows us to increase the tangential acceleration, a_t at each time step, and unwinding allows us to *decrease* the radial acceleration, a_r so we can stay within the traction circle. Since we'll still have centripetal traction available after the throttle is buried full on, we ought to be able to unwind more slowly, enabling us to stay on the track, but use it all up. In other words, we ought to look for solutions wherein k_{unwind} is larger than k, perhaps by twice.

Let's look at the first few rows of this simulation in a spreadsheet and delve into the formulas more deeply:

1	2	3	4	5	6	7	8	9	10	11	12
t	a(t) (tangential, fpsps)	v^2/r (radial, fpsps)	a(t) (radial, fpsps)	r(t) (feet)	ax(t) (fpsps)	ay(t) (fpsps)	x(t) (feet)	y(t) (feet)	vx(t) (mph)	vy(t) (mph)	v (mph)
0.00	0.00	32.00	32.00	160.00	-21.33	23.85	66.67	-74.54	36.36	32.52	48.79
0.20	1.28	31.90	30.27	169.92	-21.20	21.64	76.80	-64.41	33.46	35.66	48.90
0.40	2.56	31.59	28.54	182.30	-20.76	19.75	86.09	-53.42	30.59	38.51	49.18
0.60	3.84	31.06	26.81	197.64	-20.06	18.19	94.54	-41.64	27.79	41.12	49.63
0.80	5.12	30.32	25.08	216.59	-19.17	16.96	102.20	-29.13	25.10	43.54	50.25
0.90	5.76	29.85	24.22	227.68	-18.67	16.47	105.74	-22.62	23.80	44.69	50.63
1.00	6.40	29.33	23.35	240.01	-18.13	16.05	109.09	-15.94	22.53	45.80	51.04

[column 1]: increments by Δt each row; we actually computed with $\Delta t = 0.05\text{sec}$ and display here every fourth actual row; this is an independent column, meaning that it does not depend on data from any other column.

[column 2]: tangential acceleration, $a_t(t) = \frac{g}{2}\min\left(1, \frac{t}{k}\right)$, accounting for squeezing on the throttle up to $g/2$; depends only on column 1.

[column 3]: maximal radial acceleration, $v(t)^2 / r(t) = \sqrt{g^2 - 4a_t(t)^2}$, accounting for the traction circle; more precisely, for the upper half of the circle treated as a flattened (*oblate*) ellipse with height $g/2$; depends only on column 2.

[column 4]: radial $a_r(t) = \max\left(0, \min\left(\frac{v(t)^2}{r(t)}, g\left(1 - \frac{t}{k_{\text{unwind}}}\right)\right)\right)$, accounting for unwinding the steering wheel; in steps from the inner parentheses outwards: $g(1 - t/k_{\text{unwind}})$ slowly decreases from g as time increases from 0, but, it is never allowed to exceed v^2/r, by the **min** expression, as mandated by the traction circle, and then, never allowed to be negative, by the **max** expression, because we don't want to start turning back toward the entry straight; depends on columns 1 and 3.

[column 5]: $r(t) = v(t)^2 / a_r(t)$; just for amusement, it's interesting to calculate the instantaneous radius of a circle we could be driving if we were not accelerating tangentially; depends on columns 4 and 12, but no other columns depend on this.

[column 6]: $a_x(t) = \min\left(0, \frac{a_t v_x - a_r v_y}{v}\right)$, this just selects out the x components of both the radial and tangential accelerations, but makes sure that we never turn the wheel so much that we start going to the left. Note that the radial acceleration *always* tries to pull the car to the left, hence the minus sign (*centripetal*: see part 4 of *The Physics of Racing*); depends on columns 2, 4, 10, 11, and 12.

[column 7]: $a_y(t) = \min\left(0, \frac{a_t v_y + a_r v_x}{v}\right)$, selecting the y components, this time always pointing down the track, the way we want to go; depends on columns 2, 4, 10, 11, and 12.

[column 8]: $x(t) = x(t - \Delta t) + v_x(t)\Delta t$, just update the x coordinate by the velocity from the prior time step; depends on columns 8 (the prior row of itself) and 10.

[column 9]: $y(t) = y(t - \Delta t) + v_y(t)\Delta t$, do likewise for the y coordinate; depends on columns 9 (prior row) and 11.

[column 10]: $v_x(t) = \max(0, v_x(t - \Delta t) + a_x(t - \Delta t)\Delta t)$, for updating the x component of the velocity (but don't let it go negative, checking yet again, and, yes, this is a *hack*); depends on columns 10 (prior row) and 6.

[column 11]: $v_y(t) = v_y(t - \Delta t) + a_y(t - \Delta t)\Delta t$, likewise for the y coordinate of the velocity; depends on columns 11 and 7.

[column 12]: finally, $v = \sqrt{v_x(t)^2 + v_y(t)^2}$, depends on columns 10 and 11.

I've packed all this in an Excel spreadsheet. The spreadsheet should be in the download package for readers who acquired this document electronically.

Enough talk! Let's *drive!* Driving means playing with the values of r, k, and k_{unwind}, and possibly even Δt, to find the lowest overall time at which columns 8 and 9 show 200 or less and 650 or more, respectively. In general, "playing with" should be a sophisticated process involving hill climbing, genetic search, simulated annealing, and other fancy strategies for finding the very best values. In a computer simulation, we'd do that. However, we can do a reasonable job, for the sake of demonstration, by just tweaking the numbers by hand in the spreadsheet.

I have to admit that as I did so, I got kinesthetic feelings as if I where actually driving. When I "ran off the track," that is, picked numbers that gave me $x > 200$, I gritted my teeth and blushed. When I was still unwinding at the end, I got that panicky feeling of understeer, knowing that I wasn't going to stay on after the end of the segment, and so on.

The best values I found by hand are shown in the following table at $r = 167.5$, $k = 3.25$, and $k_{\text{unwind}} = 7.22$. That means that we take 3.25 seconds to bury the gas and 7.22 seconds to unwind the wheel. There are solutions with lower segment times, but, since we're still unwinding long after the segment is done, I reject these solutions as assuming too much about what's going on after our segment is done. With more track to work with, however, *we can find lots more time*. In fact, it's a slightly surprising fact that by taking 9 seconds to unwind at $r = 167.5$, $k = 3.25$, we lose hardly any time and stay 15 feet inside the outer edge. There is quite a bit of territory to investigate even in this simple model.

r	k	k_unwind	Best time Found	Dummy Time	Dummy-Best	Best Total Time Found
155	1.500	2.000	6.500	6.779	0.279	16.901
160	2.500	3.700	6.875	7.189	0.314	16.747
165	3.000	5.950	7.050	7.482	0.432	16.550
167.5	3.250	7.22	7.120	7.605	0.485	16.466
170	3.500	8.550	7.225	7.716	0.491	16.433
175	4.000	11.170	7.400	7.912	0.512	16.367
180	4.500	13.330	7.575	8.082	0.507	16.337
185	5.000	30.000	7.700	8.233	0.533	16.282

Since the best dummy time, with the widest possible circle, is 16.760, and the best time I found here was 16.466, **the improvement by unwinding and accelerating simultaneously is 0.294 seconds**. This is very significant. If the exit straight were longer, the improvement would be even more dramatic since it would continue to accumulate time down the straight.

Note that this does *not* involve changing the entry to the corner other than by slowing down! There is no trail braking or lifting-while-turning or other risk-taking going on at corner entry. There is a very important driving lesson, here: to go faster, it is not necessary to take risks on corner entry. It is, in fact, ***both safer and faster just to slow down on the entry***. The improved exit will follow naturally from the combination of looking far ahead and of being smooth. And that's not even fair!

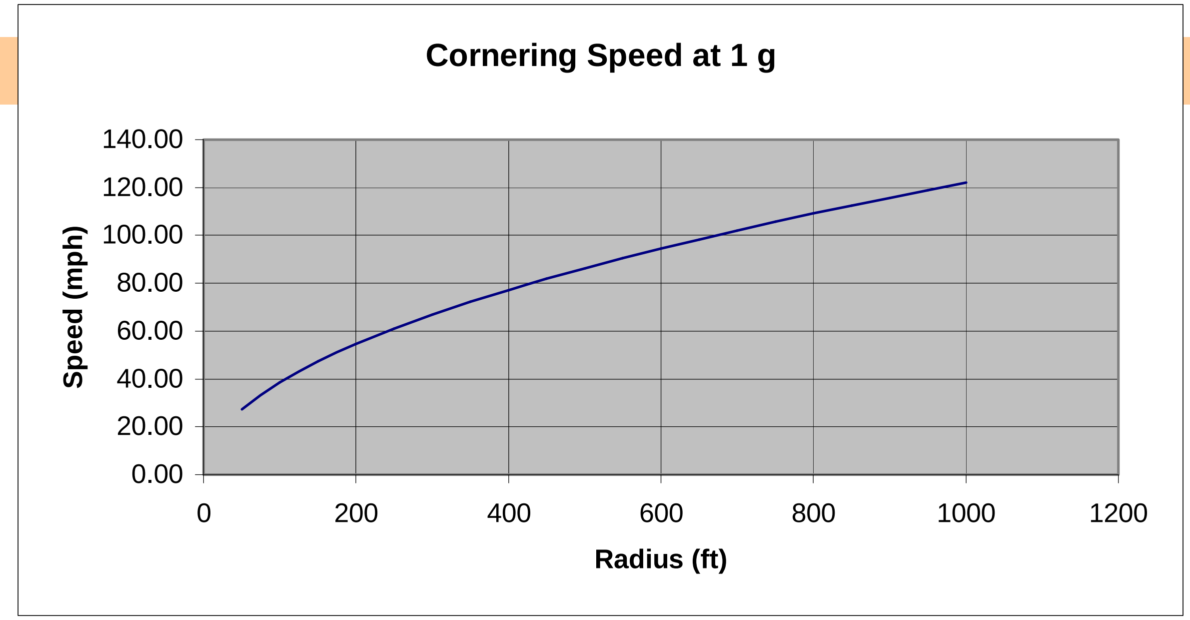
There is no guarantee that this is the best possible improvement in the model. I found these numbers by 'seat-of-the-pants' tweaking. A more systematic or algorithmic search would very likely find better ones. In other words, I was able to find almost three tenths by just driving a better line without trying very hard at all. There is another driving lesson, here: ***just driving a better line gives better times time without changing the driver's margin for error***, that is, without getting deeper into the g limits of the machine.

For the future, we can start taking more risks to get even more improvement. We can risk accelerating before the apex and we can risk deeper entry by trail braking, that is, easing off the brake and winding up the steering wheel at the same time. These maneuvers do entail more driver risk since they are new opportunities for loss of car control.

Erratum: in part 17, I wrote "By driving a line just one foot larger than the minimum, one is able to apex more than fifteen degrees later!". I should have written "... fifteen degrees *earlier*!" The point was that the tightest line does not apex until the geometric exit of the corner, and that's *way too* late. The slip-of-the-pen occurred because one is so accustomed to talking about late apexing as preferable.

Radius (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner @ 180 degrees	Time (sec) in exit chute at 1/2 g accel.	Exit speed (mph)	Total time (sec) in segment
50	27.27	311.11	238.89	1.629	3.333	3.927	4.347	74.69	13.236
100	38.57	286.11	263.89	1.799	2.816	5.554	3.753	79.51	13.922
150	47.31	260.06	389.94	2.659	2.407	6.791	5.661	109.26	17.518
200	54.63	235.06	314.94	2.147	2.073	7.842	4.675	105.79	16.737
250	60.98	211.11	338.89	2.311	1.788	8.781	2.888	92.49	15.768
300	66.80	186.11	363.89	2.481	1.521	9.619	2.715	96.42	16.337
350	72.16	161.11	388.89	2.652	1.276	10.390	2.571	100.21	16.889
400	77.14	136.11	413.89	2.822	1.048	11.107	2.449	103.85	17.426
450	81.82	111.11	438.89	2.992	0.833	11.781	2.343	107.37	17.949
500	86.24	86.11	463.89	3.163	0.630	12.418	2.249	110.78	18.461
550	90.45	61.11	488.89	3.333	0.438	13.024	2.167	114.09	18.962
600	94.48	36.11	513.89	3.504	0.253	13.603	2.093	117.30	19.453
650	98.33	11.11	538.89	3.674	0.076	14.159	2.026	120.43	19.935
700	102.05								
750	105.63								
800	109.09								
850	112.45								
900	115.71								
950	118.88								
1000	121.97								

fpspmph	1.46667	mphpfps	0.6818182
fpspmph2	2.15111	mphpfps2	0.4648760
g	32.1	fpsps	
chute	650	feet	
r_1	200	feet	
r_0	100	feet	



Start Speed (mph)	End Speed (mph)	Stopping Distance @ 1g (ft)
100	5.00	334.23
100	30.00	304.91
100	55.00	233.71
100	80.00	120.62
100	105.00	-34.34

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Time (sec) in corner after apex	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time	Combined pre-apex time and entry-chute time
150	90.00	0.00	47.31	260.06	389.94	2.659	2.407	6.791	0.000	11.857	109.261	5.661	17.518	5.661	11.857
151	73.90	14.14	47.47	259.56	376.29	2.566	2.400	6.204	0.609	11.780	108.360	5.564	17.344	6.174	11.170
152	67.38	20.00	47.63	259.06	370.94	2.529	2.393	5.977	0.859	11.758	108.025	5.519	17.278	6.378	10.899
153	62.47	24.49	47.78	258.56	366.94	2.502	2.386	5.810	1.049	11.746	107.783	5.483	17.229	6.532	10.698
154	58.41	28.28	47.94	258.06	363.65	2.479	2.379	5.674	1.207	11.739	107.590	5.451	17.190	6.659	10.532
155	54.90	31.62	48.09	257.56	360.81	2.460	2.372	5.557	1.346	11.735	107.428	5.422	17.157	6.768	10.389
160	41.81	44.72	48.86	255.06	350.21	2.388	2.336	5.136	1.878	11.738	106.864	5.300	17.038	7.178	9.860
165	32.58	54.77	49.62	252.56	342.66	2.336	2.302	4.850	2.272	11.761	106.511	5.199	16.959	7.471	9.489
170	25.38	63.25	50.37	250.06	336.69	2.296	2.268	4.634	2.596	11.793	106.267	5.108	16.901	7.704	9.198
175	19.47	70.71	51.10	247.56	331.73	2.262	2.234	4.461	2.874	11.831	106.094	5.025	16.856	7.899	8.957
180	14.48	77.46	51.83	245.06	327.48	2.233	2.201	4.318	3.121	11.873	105.971	4.948	16.821	8.069	8.752
185	10.16	83.67	52.54	242.56	323.77	2.208	2.168	4.197	3.345	11.918	105.886	4.875	16.792	8.220	8.573
190	6.38	89.44	53.25	240.06	320.49	2.185	2.136	4.092	3.551	11.965	105.831	4.805	16.770	8.356	8.414
195	3.02	94.87	53.94	237.56	317.57	2.165	2.104	4.001	3.742	12.013	105.801	4.739	16.751	8.481	8.271
200	0.00	100.00	54.63	235.06	314.94	2.147	2.073	3.921	3.921	12.062	105.792	4.675	16.737	8.596	8.141

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)
150	90.00	0.00
151	73.90	14.14
152	67.38	20.00
153	62.47	24.49
154	58.41	28.28
155	54.90	31.62
160	41.81	44.72
165	32.58	54.77
170	25.38	63.25
175	19.47	70.71
180	14.48	77.46
185	10.16	83.67
190	6.38	89.44
195	3.02	94.87
200	0.00	100.00

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Total time (sec) up to the apex
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	11.184
152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	9.500
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	8.150

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Time (sec) in corner after apex	Total time (sec) in the semi-circle	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time	Combined pre-apex time and entry-chute time
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	0.000	6.802	11.872	109.091	5.670	17.541	5.670	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	0.610	6.824	11.794	108.191	5.573	17.367	6.183	11.184

152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	0.860	6.847	11.773	107.857	5.528	17.301	6.388	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	1.051	6.869	11.761	107.615	5.492	17.252	6.542	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	1.209	6.892	11.754	107.422	5.460	17.213	6.669	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	1.348	6.914	11.750	107.260	5.430	17.180	6.779	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	1.881	7.025	11.753	106.697	5.308	17.061	7.189	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	2.276	7.134	11.776	106.345	5.207	16.982	7.482	9.500
167.5	28.78	59.16	49.92	252.36	338.48	2.308	2.295	4.743	2.444	7.188	11.791	106.212	5.160	16.951	7.605	9.346
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	2.600	7.241	11.808	106.101	5.116	16.924	7.716	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	2.879	7.347	11.846	105.928	5.033	16.879	7.912	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	3.126	7.451	11.888	105.806	4.955	16.844	8.082	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	3.350	7.554	11.933	105.721	4.882	16.815	8.233	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	3.556	7.655	11.980	105.666	4.813	16.792	8.369	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	3.748	7.755	12.028	105.636	4.746	16.774	8.494	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	3.927	7.854	12.077	105.627	4.682	16.760	8.609	8.150

r	k	k_unwind	Best time Found	Dummy Time	Dummy-Best	Best Total Time Found		Best Dummy Time	16.760
155	1.500	2.000	6.500	6.779	0.279	16.901			
160	2.500	3.700	6.875	7.189	0.314	16.747			
165	3.000	5.950	7.050	7.482	0.432	16.550			
170	3.500	8.550	7.225	7.716	0.491	16.433			
175	4.000	11.170	7.400	7.912	0.512	16.367			
180	4.500	13.330	7.575	8.082	0.507	16.337			
185	5.000	30.000	7.700	8.233	0.533	16.282			

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)
150	90.00	0.00
151	73.90	14.14
152	67.38	20.00
153	62.47	24.49
154	58.41	28.28
155	54.90	31.62
160	41.81	44.72
165	32.58	54.77
170	25.38	63.25
175	19.47	70.71
180	14.48	77.46
185	10.16	83.67
190	6.38	89.44
195	3.02	94.87
200	0.00	100.00

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Total time (sec) up to the apex
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	11.184
152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	9.500
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	8.150

Inscribed Corner Radius (ft)	Alpha (deg)	h (ft)	Cornering speed @ 1g in mph	Required Braking Distance (ft) @ 1g from 100 mph	Straight Distance (ft) prior to braking	Time (sec) in straight @ 100 mph prior to braking	Time (sec) in braking zone	Time (sec) in corner prior to apex	Time (sec) in corner after apex	Total time (sec) in the semi-circle	Time for entrance and complete corner	Exit speed from chute (mph) @ g/2 accel	Time in exit chute (sec)	Combined segment time	Combined post-apex time and exit-chute time	Combined pre-apex time and entry-chute time
150	90.00	0.00	47.24	261.11	388.89	2.652	2.418	6.802	0.000	6.802	11.872	109.091	5.670	17.541	5.670	11.872
151	73.90	14.14	47.39	260.61	375.25	2.559	2.411	6.214	0.610	6.824	11.794	108.191	5.573	17.367	6.183	11.184

152	67.38	20.00	47.55	260.11	369.89	2.522	2.404	5.987	0.860	6.847	11.773	107.857	5.528	17.301	6.388	10.912
153	62.47	24.49	47.71	259.61	365.89	2.495	2.397	5.819	1.051	6.869	11.761	107.615	5.492	17.252	6.542	10.710
154	58.41	28.28	47.86	259.11	362.60	2.472	2.390	5.682	1.209	6.892	11.754	107.422	5.460	17.213	6.669	10.544
155	54.90	31.62	48.02	258.61	359.77	2.453	2.382	5.566	1.348	6.914	11.750	107.260	5.430	17.180	6.779	10.401
160	41.81	44.72	48.79	256.11	349.17	2.381	2.347	5.144	1.881	7.025	11.753	106.697	5.308	17.061	7.189	9.872
165	32.58	54.77	49.54	253.61	341.62	2.329	2.313	4.858	2.276	7.134	11.776	106.345	5.207	16.982	7.482	9.500
167.5	28.78	59.16	49.92	252.36	338.48	2.308	2.295	4.743	2.444	7.188	11.791	106.212	5.160	16.951	7.605	9.346
170	25.38	63.25	50.29	251.11	335.64	2.288	2.278	4.641	2.600	7.241	11.808	106.101	5.116	16.924	7.716	9.208
175	19.47	70.71	51.02	248.61	330.68	2.255	2.245	4.468	2.879	7.347	11.846	105.928	5.033	16.879	7.912	8.968
180	14.48	77.46	51.75	246.11	326.43	2.226	2.212	4.325	3.126	7.451	11.888	105.806	4.955	16.844	8.082	8.762
185	10.16	83.67	52.46	243.61	322.72	2.200	2.179	4.203	3.350	7.554	11.933	105.721	4.882	16.815	8.233	8.583
190	6.38	89.44	53.16	241.11	319.45	2.178	2.147	4.099	3.556	7.655	11.980	105.666	4.813	16.792	8.369	8.424
195	3.02	94.87	53.86	238.61	316.52	2.158	2.115	4.008	3.748	7.755	12.028	105.636	4.746	16.774	8.494	8.280
200	0.00	100.00	54.55	236.11	313.89	2.140	2.083	3.927	3.927	7.854	12.077	105.627	4.682	16.760	8.609	8.150

r	k	k_unwind	Best time Found	Dummy Time	Dummy-Best	Best Total Time Found		Best Dummy Time	16.760
155	1.500	2.000	6.500	6.779	0.279	16.901			
160	2.500	3.700	6.875	7.189	0.314	16.747			
165	3.000	5.950	7.050	7.482	0.432	16.550			
170	3.500	8.550	7.225	7.716	0.491	16.433			
175	4.000	11.170	7.400	7.912	0.512	16.367			
180	4.500	13.330	7.575	8.082	0.507	16.337			
185	5.000	30.000	7.700	8.233	0.533	16.282			

t	a(t) (tangential, fpsps)	v^2/r (radial, fpsps)	a(t) (radial, fpsps)	r(t) (feet)	ax(t) (fpsps)	ay(t) (fpsps)	x(t) (feet)	y(t) (feet)	vx(t) (mph)	vy(t) (mph)	v (mph)
0.00	0.00	32.00	32.00	167.50	-15.41	28.05	48.15	-87.65	43.75	24.03	49.92
0.05	0.25	32.00	31.82	168.52	-15.71	27.67	51.32	-85.81	43.23	24.99	49.93
0.10	0.49	31.98	31.64	169.60	-16.01	27.30	54.45	-83.91	42.69	25.93	49.95
0.15	0.74	31.97	31.47	170.75	-16.29	26.93	57.54	-81.94	42.14	26.86	49.98
0.20	0.98	31.94	31.29	171.97	-16.56	26.56	60.59	-79.90	41.59	27.78	50.01
0.25	1.23	31.91	31.11	173.27	-16.82	26.20	63.60	-77.80	41.02	28.69	50.06
0.30	1.48	31.86	30.93	174.63	-17.07	25.84	66.56	-75.63	40.45	29.58	50.11
0.35	1.72	31.81	30.76	176.07	-17.30	25.48	69.49	-73.40	39.87	30.46	50.17
0.40	1.97	31.76	30.58	177.59	-17.53	25.13	72.37	-71.10	39.28	31.33	50.24
0.45	2.22	31.69	30.40	179.18	-17.74	24.78	75.20	-68.74	38.68	32.19	50.32
0.50	2.46	31.62	30.22	180.86	-17.95	24.44	78.00	-66.32	38.08	33.03	50.41
0.55	2.71	31.54	30.04	182.61	-18.14	24.10	80.74	-63.83	37.46	33.87	50.50
0.60	2.95	31.45	29.87	184.44	-18.32	23.77	83.45	-61.29	36.85	34.69	50.60
0.65	3.20	31.35	29.69	186.36	-18.50	23.44	86.10	-58.69	36.22	35.50	50.72
0.70	3.45	31.25	29.51	188.37	-18.66	23.12	88.71	-56.02	35.59	36.30	50.84
0.75	3.69	31.14	29.33	190.46	-18.81	22.81	91.28	-53.30	34.95	37.09	50.96
0.80	3.94	31.02	29.16	192.64	-18.96	22.50	93.79	-50.53	34.31	37.86	51.10
0.85	4.18	30.89	28.98	194.92	-19.10	22.19	96.26	-47.69	33.67	38.63	51.24
0.90	4.43	30.75	28.80	197.29	-19.22	21.90	98.68	-44.81	33.02	39.39	51.39
0.95	4.68	30.60	28.62	199.75	-19.35	21.61	101.05	-41.86	32.36	40.13	51.55
1.00	4.92	30.45	28.44	202.32	-19.46	21.32	103.38	-38.87	31.70	40.87	51.72
1.05	5.17	30.28	28.27	204.99	-19.56	21.05	105.66	-35.82	31.04	41.60	51.90
1.10	5.42	30.11	28.09	207.76	-19.66	20.78	107.88	-32.71	30.37	42.31	52.09
1.15	5.66	29.93	27.91	210.64	-19.75	20.52	110.06	-29.56	29.70	43.02	52.28
1.20	5.91	29.74	27.73	213.63	-19.84	20.26	112.19	-26.35	29.03	43.72	52.48
1.25	6.15	29.54	27.56	216.73	-19.92	20.01	114.27	-23.09	28.35	44.41	52.69
1.30	6.40	29.33	27.38	219.94	-19.99	19.77	116.30	-19.79	27.67	45.09	52.91
1.35	6.65	29.11	27.20	223.28	-20.05	19.54	118.28	-16.43	26.99	45.77	53.13
1.40	6.89	28.88	27.02	226.74	-20.11	19.32	120.21	-13.03	26.31	46.44	53.37
1.45	7.14	28.64	26.84	230.32	-20.17	19.10	122.09	-9.57	25.62	47.09	53.61
1.50	7.38	28.39	26.67	234.03	-20.22	18.89	123.91	-6.07	24.93	47.74	53.86
1.55	7.63	28.13	26.49	237.88	-20.26	18.69	125.69	-2.52	24.24	48.39	54.12
1.60	7.88	27.85	26.31	241.86	-20.31	18.49	127.42	1.07	23.55	49.03	54.39
1.65	8.12	27.57	26.13	245.98	-20.34	18.31	129.10	4.71	22.86	49.66	54.67
1.70	8.37	27.27	25.96	250.25	-20.37	18.13	130.72	8.40	22.17	50.28	54.95
1.75	8.62	26.96	25.78	254.66	-20.40	17.96	132.30	12.13	21.47	50.90	55.24
1.80	8.86	26.64	25.60	259.23	-20.43	17.79	133.82	15.91	20.78	51.51	55.54
1.85	9.11	26.31	25.42	263.96	-20.45	17.64	135.29	19.73	20.08	52.12	55.85
1.90	9.35	25.96	25.24	268.84	-20.47	17.49	136.71	23.60	19.38	52.72	56.17
1.95	9.60	25.60	25.07	273.90	-20.48	17.35	138.08	27.51	18.69	53.32	56.49
2.00	9.85	25.22	24.89	279.12	-20.49	17.22	139.40	31.46	17.99	53.91	56.83
2.05	10.09	24.83	24.71	284.52	-20.50	17.09	140.67	35.46	17.29	54.49	57.17
2.10	10.34	24.42	24.42	291.42	-20.40	16.94	141.89	39.50	16.59	55.08	57.52
2.15	10.58	24.00	24.00	300.30	-20.17	16.77	143.05	43.58	15.90	55.65	57.88
2.20	10.83	23.55	23.55	309.84	-19.91	16.60	144.17	47.70	15.21	56.23	58.25
2.25	11.08	23.09	23.09	320.12	-19.63	16.45	145.23	51.87	14.53	56.79	58.62
2.30	11.32	22.61	22.61	331.24	-19.32	16.32	146.25	56.07	13.86	57.35	59.00
2.35	11.57	22.10	22.10	343.30	-18.98	16.19	147.22	60.32	13.20	57.91	59.39

parameters		
r0	100	feet
r_1	200	feet
r	167.5	feet
alpha	28.78	degrees
h	59.16	feet
w	100	feet
k	3.25	seconds
v0	49.92	mph
amax	16	fpsps
deltat	0.05	seconds
g	32	fpsps
k_unwind	9.00	seconds

2.40	11.82	21.58	21.58	356.43	-18.62	16.08	148.14	64.61	12.55	58.46	59.79
2.45	12.06	21.03	21.03	370.78	-18.22	15.99	149.01	68.93	11.92	59.01	60.20
2.50	12.31	20.45	20.45	386.55	-17.79	15.90	149.84	73.30	11.30	59.55	60.62
2.55	12.55	19.84	19.84	403.98	-17.33	15.83	150.63	77.71	10.69	60.10	61.04
2.60	12.80	19.20	19.20	423.36	-16.84	15.78	151.37	82.15	10.10	60.64	61.47
2.65	13.05	18.53	18.53	445.08	-16.30	15.74	152.07	86.64	9.53	61.17	61.91
2.70	13.29	17.81	17.81	469.63	-15.71	15.72	152.72	91.17	8.97	61.71	62.36
2.75	13.54	17.05	17.05	497.70	-15.08	15.71	153.34	95.73	8.44	62.25	62.82
2.80	13.78	16.25	16.25	530.20	-14.39	15.71	153.92	100.33	7.92	62.78	63.28
2.85	14.03	15.38	15.38	568.45	-13.64	15.73	154.47	104.98	7.43	63.32	63.75
2.90	14.28	14.45	14.45	614.37	-12.81	15.76	154.98	109.66	6.97	63.85	64.23
2.95	14.52	13.43	13.43	671.01	-11.89	15.80	155.46	114.38	6.53	64.39	64.72
3.00	14.77	12.31	12.31	743.39	-10.87	15.86	155.91	119.14	6.12	64.93	65.22
3.05	15.02	11.05	11.05	840.70	-9.70	15.93	156.33	123.95	5.75	65.47	65.72
3.10	15.26	9.61	9.61	982.08	-8.33	16.00	156.73	128.79	5.42	66.01	66.24
3.15	15.51	7.88	7.88	1217.01	-6.66	16.07	157.10	133.67	5.14	66.56	66.76
3.20	15.75	5.59	5.59	1741.72	-4.43	16.12	157.46	138.59	4.91	67.11	67.29
3.25	16.00	0.00	0.00	########	0.00	15.96	157.81	143.55	4.76	67.66	67.82
3.30	16.00	0.00	0.00	#DIV/0!	0.00	15.96	158.16	148.55	4.76	68.20	68.37
3.35	16.00	0.00	0.00	#DIV/0!	0.00	15.96	158.51	153.59	4.76	68.74	68.91
3.40	16.00	0.00	0.00	#DIV/0!	0.00	15.96	158.86	158.67	4.76	69.29	69.45
3.45	16.00	0.00	0.00	#DIV/0!	0.00	15.96	159.21	163.79	4.76	69.83	69.99
3.50	16.00	0.00	0.00	#DIV/0!	0.00	15.96	159.56	168.96	4.76	70.38	70.54
3.55	16.00	0.00	0.00	#DIV/0!	0.00	15.96	159.91	174.16	4.76	70.92	71.08
3.60	16.00	0.00	0.00	#DIV/0!	0.00	15.96	160.26	179.40	4.76	71.47	71.62
3.65	16.00	0.00	0.00	#DIV/0!	0.00	15.97	160.61	184.68	4.76	72.01	72.17
3.70	16.00	0.00	0.00	#DIV/0!	0.00	15.97	160.95	190.00	4.76	72.55	72.71
3.75	16.00	0.00	0.00	#DIV/0!	0.00	15.97	161.30	195.36	4.76	73.10	73.25
3.80	16.00	0.00	0.00	#DIV/0!	0.00	15.97	161.65	200.76	4.76	73.64	73.80
3.85	16.00	0.00	0.00	#DIV/0!	0.00	15.97	162.00	206.20	4.76	74.19	74.34
3.90	16.00	0.00	0.00	#DIV/0!	0.00	15.97	162.35	211.68	4.76	74.73	74.88
3.95	16.00	0.00	0.00	#DIV/0!	0.00	15.97	162.70	217.20	4.76	75.28	75.43
4.00	16.00	0.00	0.00	#DIV/0!	0.00	15.97	163.05	222.76	4.76	75.82	75.97
4.05	16.00	0.00	0.00	#DIV/0!	0.00	15.97	163.40	228.36	4.76	76.36	76.51
4.10	16.00	0.00	0.00	#DIV/0!	0.00	15.97	163.75	234.00	4.76	76.91	77.06
4.15	16.00	0.00	0.00	#DIV/0!	0.00	15.97	164.10	239.68	4.76	77.45	77.60
4.20	16.00	0.00	0.00	#DIV/0!	0.00	15.97	164.45	245.40	4.76	78.00	78.14
4.25	16.00	0.00	0.00	#DIV/0!	0.00	15.97	164.79	251.16	4.76	78.54	78.69
4.30	16.00	0.00	0.00	#DIV/0!	0.00	15.97	165.14	256.96	4.76	79.09	79.23
4.35	16.00	0.00	0.00	#DIV/0!	0.00	15.97	165.49	262.80	4.76	79.63	79.77
4.40	16.00	0.00	0.00	#DIV/0!	0.00	15.97	165.84	268.68	4.76	80.17	80.32
4.45	16.00	0.00	0.00	#DIV/0!	0.00	15.97	166.19	274.60	4.76	80.72	80.86
4.50	16.00	0.00	0.00	#DIV/0!	0.00	15.97	166.54	280.56	4.76	81.26	81.40
4.55	16.00	0.00	0.00	#DIV/0!	0.00	15.97	166.89	286.56	4.76	81.81	81.95
4.60	16.00	0.00	0.00	#DIV/0!	0.00	15.97	167.24	292.60	4.76	82.35	82.49
4.65	16.00	0.00	0.00	#DIV/0!	0.00	15.97	167.59	298.67	4.76	82.90	83.03
4.70	16.00	0.00	0.00	#DIV/0!	0.00	15.97	167.94	304.79	4.76	83.44	83.58
4.75	16.00	0.00	0.00	#DIV/0!	0.00	15.97	168.29	310.95	4.76	83.99	84.12
4.80	16.00	0.00	0.00	#DIV/0!	0.00	15.97	168.64	317.15	4.76	84.53	84.67
4.85	16.00	0.00	0.00	#DIV/0!	0.00	15.98	168.98	323.39	4.76	85.08	85.21
4.90	16.00	0.00	0.00	#DIV/0!	0.00	15.98	169.33	329.67	4.76	85.62	85.75

4.95	16.00	0.00	0.00	#DIV/0!	0.00	15.98	169.68	335.99	4.76	86.17	86.30
5.00	16.00	0.00	0.00	#DIV/0!	0.00	15.98	170.03	342.35	4.76	86.71	86.84
5.05	16.00	0.00	0.00	#DIV/0!	0.00	15.98	170.38	348.75	4.76	87.25	87.38
5.10	16.00	0.00	0.00	#DIV/0!	0.00	15.98	170.73	355.18	4.76	87.80	87.93
5.15	16.00	0.00	0.00	#DIV/0!	0.00	15.98	171.08	361.66	4.76	88.34	88.47
5.20	16.00	0.00	0.00	#DIV/0!	0.00	15.98	171.43	368.18	4.76	88.89	89.02
5.25	16.00	0.00	0.00	#DIV/0!	0.00	15.98	171.78	374.74	4.76	89.43	89.56
5.30	16.00	0.00	0.00	#DIV/0!	0.00	15.98	172.13	381.34	4.76	89.98	90.10
5.35	16.00	0.00	0.00	#DIV/0!	0.00	15.98	172.48	387.98	4.76	90.52	90.65
5.40	16.00	0.00	0.00	#DIV/0!	0.00	15.98	172.82	394.65	4.76	91.07	91.19
5.45	16.00	0.00	0.00	#DIV/0!	0.00	15.98	173.17	401.37	4.76	91.61	91.74
5.50	16.00	0.00	0.00	#DIV/0!	0.00	15.98	173.52	408.13	4.76	92.16	92.28
5.55	16.00	0.00	0.00	#DIV/0!	0.00	15.98	173.87	414.93	4.76	92.70	92.82
5.60	16.00	0.00	0.00	#DIV/0!	0.00	15.98	174.22	421.77	4.76	93.25	93.37
5.65	16.00	0.00	0.00	#DIV/0!	0.00	15.98	174.57	428.64	4.76	93.79	93.91
5.70	16.00	0.00	0.00	#DIV/0!	0.00	15.98	174.92	435.56	4.76	94.34	94.46
5.75	16.00	0.00	0.00	#DIV/0!	0.00	15.98	175.27	442.52	4.76	94.88	95.00
5.80	16.00	0.00	0.00	#DIV/0!	0.00	15.98	175.62	449.52	4.76	95.42	95.54
5.85	16.00	0.00	0.00	#DIV/0!	0.00	15.98	175.97	456.56	4.76	95.97	96.09
5.90	16.00	0.00	0.00	#DIV/0!	0.00	15.98	176.32	463.63	4.76	96.51	96.63
5.95	16.00	0.00	0.00	#DIV/0!	0.00	15.98	176.67	470.75	4.76	97.06	97.18
6.00	16.00	0.00	0.00	#DIV/0!	0.00	15.98	177.01	477.91	4.76	97.60	97.72
6.05	16.00	0.00	0.00	#DIV/0!	0.00	15.98	177.36	485.11	4.76	98.15	98.26
6.10	16.00	0.00	0.00	#DIV/0!	0.00	15.98	177.71	492.34	4.76	98.69	98.81
6.15	16.00	0.00	0.00	#DIV/0!	0.00	15.98	178.06	499.62	4.76	99.24	99.35
6.20	16.00	0.00	0.00	#DIV/0!	0.00	15.98	178.41	506.94	4.76	99.78	99.90
6.25	16.00	0.00	0.00	#DIV/0!	0.00	15.98	178.76	514.30	4.76	100.33	100.44
6.30	16.00	0.00	0.00	#DIV/0!	0.00	15.98	179.11	521.69	4.76	100.87	100.99
6.35	16.00	0.00	0.00	#DIV/0!	0.00	15.98	179.46	529.13	4.76	101.42	101.53
6.40	16.00	0.00	0.00	#DIV/0!	0.00	15.98	179.81	536.61	4.76	101.96	102.07
6.45	16.00	0.00	0.00	#DIV/0!	0.00	15.98	180.16	544.13	4.76	102.51	102.62
6.50	16.00	0.00	0.00	#DIV/0!	0.00	15.98	180.51	551.68	4.76	103.05	103.16
6.55	16.00	0.00	0.00	#DIV/0!	0.00	15.98	180.85	559.28	4.76	103.60	103.71
6.60	16.00	0.00	0.00	#DIV/0!	0.00	15.98	181.20	566.92	4.76	104.14	104.25
6.65	16.00	0.00	0.00	#DIV/0!	0.00	15.98	181.55	574.59	4.76	104.69	104.80
6.70	16.00	0.00	0.00	#DIV/0!	0.00	15.98	181.90	582.31	4.76	105.23	105.34
6.75	16.00	0.00	0.00	#DIV/0!	0.00	15.98	182.25	590.07	4.76	105.78	105.88
6.80	16.00	0.00	0.00	#DIV/0!	0.00	15.98	182.60	597.87	4.76	106.32	106.43
6.85	16.00	0.00	0.00	#DIV/0!	0.00	15.98	182.95	605.70	4.76	106.87	106.97
6.90	16.00	0.00	0.00	#DIV/0!	0.00	15.98	183.30	613.58	4.76	107.41	107.52
6.95	16.00	0.00	0.00	#DIV/0!	0.00	15.98	183.65	621.50	4.76	107.96	108.06
7.00	16.00	0.00	0.00	#DIV/0!	0.00	15.98	184.00	629.45	4.76	108.50	108.61
7.05	16.00	0.00	0.00	#DIV/0!	0.00	15.98	184.35	637.45	4.76	109.05	109.15
7.10	16.00	0.00	0.00	#DIV/0!	0.00	15.98	184.70	645.49	4.76	109.59	109.69
7.15	16.00	0.00	0.00	#DIV/0!	0.00	15.99	185.04	653.56	4.76	110.14	110.24
7.20	16.00	0.00	0.00	#DIV/0!	0.00	15.99	185.39	661.68	4.76	110.68	110.78
7.25	16.00	0.00	0.00	#DIV/0!	0.00	15.99	185.74	669.84	4.76	111.23	111.33
7.30	16.00	0.00	0.00	#DIV/0!	0.00	15.99	186.09	678.03	4.76	111.77	111.87
7.35	16.00	0.00	0.00	#DIV/0!	0.00	15.99	186.44	686.27	4.76	112.32	112.42
7.40	16.00	0.00	0.00	#DIV/0!	0.00	15.99	186.79	694.55	4.76	112.86	112.96
7.45	16.00	0.00	0.00	#DIV/0!	0.00	15.99	187.14	702.86	4.76	113.41	113.51

7.50	16.00	0.00	0.00	#DIV/0!	0.00	15.99	187.49	711.22	4.76	113.95	114.05
7.55	16.00	0.00	0.00	#DIV/0!	0.00	15.99	187.84	719.62	4.76	114.50	114.59
7.60	16.00	0.00	0.00	#DIV/0!	0.00	15.99	188.19	728.05	4.76	115.04	115.14
7.65	16.00	0.00	0.00	#DIV/0!	0.00	15.99	188.54	736.53	4.76	115.59	115.68
7.70	16.00	0.00	0.00	#DIV/0!	0.00	15.99	188.88	745.04	4.76	116.13	116.23
7.75	16.00	0.00	0.00	#DIV/0!	0.00	15.99	189.23	753.60	4.76	116.68	116.77
7.80	16.00	0.00	0.00	#DIV/0!	0.00	15.99	189.58	762.20	4.76	117.22	117.32
7.85	16.00	0.00	0.00	#DIV/0!	0.00	15.99	189.93	770.83	4.76	117.77	117.86
7.90	16.00	0.00	0.00	#DIV/0!	0.00	15.99	190.28	779.51	4.76	118.31	118.41
7.95	16.00	0.00	0.00	#DIV/0!	0.00	15.99	190.63	788.22	4.76	118.86	118.95
8.00	16.00	0.00	0.00	#DIV/0!	0.00	15.99	190.98	796.98	4.76	119.40	119.50
8.05	16.00	0.00	0.00	#DIV/0!	0.00	15.99	191.33	805.78	4.76	119.95	120.04
8.10	16.00	0.00	0.00	#DIV/0!	0.00	15.99	191.68	814.61	4.76	120.49	120.59
8.15	16.00	0.00	0.00	#DIV/0!	0.00	15.99	192.03	823.49	4.76	121.04	121.13
8.20	16.00	0.00	0.00	#DIV/0!	0.00	15.99	192.38	832.40	4.76	121.58	121.67
8.25	16.00	0.00	0.00	#DIV/0!	0.00	15.99	192.73	841.36	4.76	122.13	122.22
8.30	16.00	0.00	0.00	#DIV/0!	0.00	15.99	193.07	850.36	4.76	122.67	122.76
8.35	16.00	0.00	0.00	#DIV/0!	0.00	15.99	193.42	859.39	4.76	123.22	123.31
8.40	16.00	0.00	0.00	#DIV/0!	0.00	15.99	193.77	868.47	4.76	123.76	123.85
8.45	16.00	0.00	0.00	#DIV/0!	0.00	15.99	194.12	877.58	4.76	124.31	124.40
8.50	16.00	0.00	0.00	#DIV/0!	0.00	15.99	194.47	886.74	4.76	124.85	124.94
8.55	16.00	0.00	0.00	#DIV/0!	0.00	15.99	194.82	895.94	4.76	125.40	125.49
8.60	16.00	0.00	0.00	#DIV/0!	0.00	15.99	195.17	905.17	4.76	125.94	126.03
8.65	16.00	0.00	0.00	#DIV/0!	0.00	15.99	195.52	914.45	4.76	126.49	126.58
8.70	16.00	0.00	0.00	#DIV/0!	0.00	15.99	195.87	923.76	4.76	127.03	127.12
8.75	16.00	0.00	0.00	#DIV/0!	0.00	15.99	196.22	933.12	4.76	127.58	127.67
8.80	16.00	0.00	0.00	#DIV/0!	0.00	15.99	196.57	942.51	4.76	128.12	128.21
8.85	16.00	0.00	0.00	#DIV/0!	0.00	15.99	196.91	951.95	4.76	128.67	128.75
8.90	16.00	0.00	0.00	#DIV/0!	0.00	15.99	197.26	961.43	4.76	129.21	129.30
8.95	16.00	0.00	0.00	#DIV/0!	0.00	15.99	197.61	970.94	4.76	129.76	129.84
9.00	16.00	0.00	0.00	#DIV/0!	0.00	15.99	197.96	980.50	4.76	130.30	130.39
9.05	16.00	0.00	0.00	#DIV/0!	0.00	15.99	198.31	990.09	4.76	130.85	130.93
9.10	16.00	0.00	0.00	#DIV/0!	0.00	15.99	198.66	999.73	4.76	131.39	131.48
9.15	16.00	0.00	0.00	#DIV/0!	0.00	15.99	199.01	1009.40	4.76	131.94	132.02
9.20	16.00	0.00	0.00	#DIV/0!	0.00	15.99	199.36	1019.12	4.76	132.48	132.57
9.25	16.00	0.00	0.00	#DIV/0!	0.00	15.99	199.71	1028.87	4.76	133.03	133.11
9.30	16.00	0.00	0.00	#DIV/0!	0.00	15.99	200.06	1038.67	4.76	133.57	133.66
9.35	16.00	0.00	0.00	#DIV/0!	0.00	15.99	200.41	1048.50	4.76	134.12	134.20
9.40	16.00	0.00	0.00	#DIV/0!	0.00	15.99	200.76	1058.38	4.76	134.66	134.75
9.45	16.00	0.00	0.00	#DIV/0!	0.00	15.99	201.10	1068.29	4.76	135.21	135.29
9.50	16.00	0.00	0.00	#DIV/0!	0.00	15.99	201.45	1078.25	4.76	135.75	135.84
9.55	16.00	0.00	0.00	#DIV/0!	0.00	15.99	201.80	1088.24	4.76	136.30	136.38
9.60	16.00	0.00	0.00	#DIV/0!	0.00	15.99	202.15	1098.28	4.76	136.84	136.93
9.65	16.00	0.00	0.00	#DIV/0!	0.00	15.99	202.50	1108.36	4.76	137.39	137.47
9.70	16.00	0.00	0.00	#DIV/0!	0.00	15.99	202.85	1118.47	4.76	137.93	138.02
9.75	16.00	0.00	0.00	#DIV/0!	0.00	15.99	203.20	1128.63	4.76	138.48	138.56
9.80	16.00	0.00	0.00	#DIV/0!	0.00	15.99	203.55	1138.82	4.76	139.02	139.11
9.85	16.00	0.00	0.00	#DIV/0!	0.00	15.99	203.90	1149.06	4.76	139.57	139.65
9.90	16.00	0.00	0.00	#DIV/0!	0.00	15.99	204.25	1159.33	4.76	140.11	140.20
9.95	16.00	0.00	0.00	#DIV/0!	0.00	15.99	204.60	1169.65	4.76	140.66	140.74
10.00	16.00	0.00	0.00	#DIV/0!	0.00	15.99	204.94	1180.00	4.76	141.20	141.28

10.05	16.00	0.00	0.00	#DIV/0!	0.00	15.99	205.29	1190.40	4.76	141.75	141.83
10.10	16.00	0.00	0.00	#DIV/0!	0.00	15.99	205.64	1200.83	4.76	142.29	142.37
10.15	16.00	0.00	0.00	#DIV/0!	0.00	15.99	205.99	1211.31	4.76	142.84	142.92
10.20	16.00	0.00	0.00	#DIV/0!	0.00	15.99	206.34	1221.82	4.76	143.39	143.46
10.25	16.00	0.00	0.00	#DIV/0!	0.00	15.99	206.69	1232.38	4.76	143.93	144.01
10.30	16.00	0.00	0.00	#DIV/0!	0.00	15.99	207.04	1242.97	4.76	144.48	144.55
10.35	16.00	0.00	0.00	#DIV/0!	0.00	15.99	207.39	1253.60	4.76	145.02	145.10
10.40	16.00	0.00	0.00	#DIV/0!	0.00	15.99	207.74	1264.28	4.76	145.57	145.64
10.45	16.00	0.00	0.00	#DIV/0!	0.00	15.99	208.09	1274.99	4.76	146.11	146.19
10.50	16.00	0.00	0.00	#DIV/0!	0.00	15.99	208.44	1285.75	4.76	146.66	146.73
10.55	16.00	0.00	0.00	#DIV/0!	0.00	15.99	208.79	1296.54	4.76	147.20	147.28
10.60	16.00	0.00	0.00	#DIV/0!	0.00	15.99	209.13	1307.38	4.76	147.75	147.82
10.65	16.00	0.00	0.00	#DIV/0!	0.00	15.99	209.48	1318.25	4.76	148.29	148.37
10.70	16.00	0.00	0.00	#DIV/0!	0.00	15.99	209.83	1329.17	4.76	148.84	148.91
10.75	16.00	0.00	0.00	#DIV/0!	0.00	15.99	210.18	1340.12	4.76	149.38	149.46
10.80	16.00	0.00	0.00	#DIV/0!	0.00	15.99	210.53	1351.12	4.76	149.93	150.00
10.85	16.00	0.00	0.00	#DIV/0!	0.00	15.99	210.88	1362.15	4.76	150.47	150.55
10.90	16.00	0.00	0.00	#DIV/0!	0.00	15.99	211.23	1373.23	4.76	151.02	151.09
10.95	16.00	0.00	0.00	#DIV/0!	0.00	15.99	211.58	1384.34	4.76	151.56	151.64